Large and Huge Monoidal Algebras for the Quantum Mechanical Oscillator

Publisher, author and contact for feedback is (9.7.1999):
Haiko Schmarsow
Birkenweg 6
D-65623 Hahnstätten
haiko@schmarsow.de
http://www.schmarsow.de

Written in LyX 1.0.1, converted to PDF with Adobe Acrobat 4.0.

ISBN 3-00-004696-8

Abstract

The eigenvalue problem of the quantum mechanical harmonic oscillator can be solved by applying a new method. The same method, used together with a special case of the Baker-Campbell-Hausdorff series formula, yields the Lie group of the oscillator algebra. The method uses the Heisenberg commutation relation and an extension of linear combinations to infinitely many summands. This extension of the notion of a linear combination gives the basic concept of a so-called *reduced basis*. That kind of basis is not covered by the notions of an algebraic basis or a topological basis. Central notions are also those of a commutative field, namely the fields of the real and complex numbers, and a set that generates a free monoid. The free monoid is the underlying structure of the associative unital multiplication in the emerging algebras. Quotient processes then produce algebras in which are realized the Heisenberg- and oscillator commutation relations. Then these enveloping algebras are extended to large and so-called *huge* versions. In one large version the eigenvalue problem of the harmonic oscillator is solved. In the other huge version the Lie group of the (harmonic) oscillator algebra can be calculated by exponentiation of generating elements.

Introduction

Observables in quantum mechanics usually are written as operators on a Hilbert space. Then, numerical results are predicted by solving eigenproblems of the operators involved and by considering appropriate scalar products on the Hilbert space. There the operator algebras are representations of algebraic relations between physical observables. In the last chapter of this text relevant physical information for the special case of the harmonic oscillator is gathered without the use of representations. In the same way, elements of the Heisenberg- and oscillator Lie algebra can be exponentiated in a *huge algebra*, giving a parametrization of the respective Lie groups.

Basic notions in that context are linear combinations of monoid elements and their extensions to sums of infinitely many summands. The latter gives rise to the concept of the so-called *reduced basis* which can be identified as an algebraic or topological basis only in special cases.

Algebras can be constructed explicitly from a commutative field and a set: The elements of that given set can be used to construct a free monoid which furnishes the essence of the multiplicative structure of the algebra to be constructed. — Copies of the field indexed by the elements of the free monoid make up a direct sum that can be given the structure of an associative unital algebra. In these algebras, the free monoid, or rather the set of the coordinates of the monoid elements, is an algebraic basis of the vector space structure. — The product space, given by copies of the field indexed by monoid elements, can be recognized as a large algebra of the associative unital kind. The product topology makes this large algebra topological, and only if the field is Hausdorff the set of coordinates of the free monoid becomes a so-called *reduced basis*. Since the coordinate notation would involve too much writing in the calculations, a notation is used that treats monoid elements as basic vectors. — A smaller algebra is the algebra of formal power series (also called *series algebra*) which becomes a topological algebra with respect to a so-called *product box topology*. This kind of algebra, considered over a discretely topologized field, is called Magnus algebra.

In all the algebras mentioned so far there are no algebraic relations. Two strategies may be thought of to construct enlarged algebras with relations, especially with commutator relations: Consider the large algebra of a free monoid that has been mentioned previously and try to construct a quotient using the given relations. In this special case this strategy doesn't succeed because the multiplicative identity is contained inside the ideal defining the quotient, so the quotient space collapses to a point.

The other strategy takes the direct sum space instead of the product space, and introduces the given relations by a nondegenerate quotient process. Enlarging this quotient space are obtained large and huge extensions of the initial quotient algebra. In these extensions can be calculated the solutions of the eigenvalue problem of the harmonic oscillator and the Lie group of the harmonic oscillator Lie algebra. The quotient algebras used here are not universal enveloping algebras because those algebras are unnecessarily large.

The following list gives a short characterization of the contents of each chapter:

The Chapter “Product Spaces” discusses differences of the product and box topologies, especially in topological groups and vector spaces. Some of this information is used in Chapter “Algebras of Monoids” in the discussion of the spaces of formal power series.

The Chapter “Linear Sums in Hausdorff Topological Vector Spaces” introduces the concepts of a *linear sum of a family* and that of a *linear sum of a set,* which both are defined in Hausdorff topological vector spaces. The *linear sum of a family* of elements extends the idea of a linear combination to sums of (possibly) infinitely many elements. The *linear sum of a generating set* is the set of all linear sums which can be formed with the elements of that generating set. In this aspect the notion of the linear sum resembles that of the linear span.

The Chapter “Algebraic-, Topological- and Reduced Basis” extends the notion of a linear sum to that of a *reduced basis*. This is done by introducing the idea of the so-called *reduced freedom*. An example shows the notion of a reduced basis existing independently from those of an algebraic- or topological basis.

The Chapter “Algebras of Monoids” shows different associative unital algebras constructed from a commutative Hausdorff topological field and a free monoid. Only the structures of a direct sum and product set are used to construct the direct sum space, the product space and the space of formal power series of a free monoid.

The Chapter “Ideals and Envelopes in Algebras” considers ideals in the spaces of the previous chapter, an example makes these large algebras unsuited for a quotient process that produces commutation relations.

The Chapter “Monoidal Quotient Algebras” defines quotient algebras (Weyl-like algebras) using the universal enveloping algebras of Lie algebras and gives the relevant algebraic bases. This process is exemplified for the Heisenberg Lie algebra and the oscillator Lie algebra. The notion of an enveloping algebra is used in a more general sense than that of a universal enveloping algebra of a Lie algebra.

The Chapter “Large and Huge Monoidal Algebras” extends these enveloping algebras and presents some calculations in these large and *huge* extensions. Thereby the eigenproblem of the harmonic oscillator in quantum mechanics is solved on an algebraic level and a parametrization of the oscillator Lie group is given by exponentiating elements of the huge associative oscillator algebra.

The Statement 53 (on page 54) about the centered enveloping algebra and the Statement 63 (on page 62) about the extension of algebras are helpers for the calculations and present no structural analysis. The results in the Supplement are not multiply cross-checked.

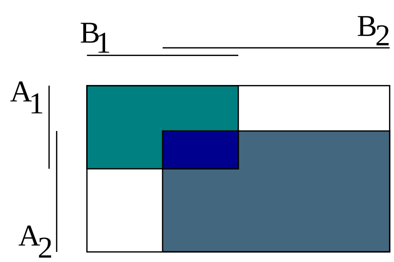
Contents

Product Spaces

This first chapter presents some elementary results and ideas about product spaces. Boxes and generalized direct sums are introduced and some of their set-theoretic and topological properties are investigated. And there is discussed the box topology on product groups and product vector spaces.

Boxes

For an index set J, consider the notion of a product $\prod_{j\in J} X_j$ of a family $(X_j)_{j\in J}$ of sets as introduced for example in [EII.32 §5.3 Def. 1] or [WGT.52 Def. 8.1]. Then consider a family of subsets $(A_j)_{j\in J}$ $(\forall j\,(j\in J\Rightarrow A_j\subset X_j))$; the subset $B\subset\prod_{j\in J} X_j$ of the product space, that is isomorphic to the product $\prod_{j\in J} A_j$, is called *box*. For ease of notation, the box B and the defining product space $\prod_{j\in J} A_j$ are identified. Not every subset of a product space is a box: In the product space $\{0,1\}\times\{a,b\}$ the subset $\{(0,a),(1,b)\}$ is not a box. The results below also can be reached heuristically by considering rectangles in a two-dimensional surface:



1 **STATEMENT.** ***(Intersections of Boxes)****: In any product space the intersection of (two) boxes is a box, which is the product set of the intersection of their factor sets:*

$$\forall J\left(J\,\mathrm{set}\Rightarrow\forall\,(X_j)_{j\in J}\Big(\quad\forall j\,(j\in J\Rightarrow X_j\,\mathrm{set\,and}\,\forall\,(A_j,\,B_j\subset X_j))\Rightarrow \prod_{j\in J} A_j\bigcap\prod_{j\in J} B_j=\prod_{j\in J}(A_j\bigcap B_j)\quad\Big)\right)$$

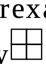
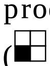
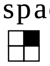
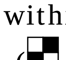
Proof Indication: "$\subset$": If the set $\prod_{j\in J} A_j\cap\prod_{j\in J} B_j$ is empty, then the statement is true, but if the mentioned set is not empty, then consider one of its elements $x\in\prod_{j\in J} A_j\cap\prod_{j\in J} B_j$. The definition of the intersection makes the previous formula equivalent to $x\in\prod_{j\in J} A_j$ AND $x\in\prod_{j\in J} B_j$. Since the element x is an element of the product space $\prod_{j\in J} X_j$, write the element $x=(x_j)_{j\in J}$. The two previous sentences give $\forall j\,(j \in J \Rightarrow x_j \in A_j)$ AND $\forall j\,(j \in J \Rightarrow x_j \in B_j)$. These two results are equivalent to $\forall j\,(j \in J \Rightarrow x_j \in A_j \cap B_j)$, which in turn, by the definition of the product space, is equivalent to $x = (x_j)_{j\in J} \in \prod_{j\in J}(A_j \cap B_j)$. Since the element x was chosen arbitrarily out of the initial set the desired subset relation is established.

"$\supset$": (Basically a variation of the above, but a clearly distinct one!) If the product set $\prod_{j\in J}(A_j \cap B_j)$ is empty , then the statement is true, but if the mentioned set is not empty, then consider one of its elements $z \in \prod_{j\in J}(A_j \cap B_j)$. Since the element z is an element of the product space $\prod_{j\in J} X_j$, write the element $z = (z_j)_{j\in J}$. The two previous sentences give $\forall j\,(j \in J \Rightarrow z_j \in A_j \cap B_j)$. The definition of the intersection makes the previous formula equivalent to $\forall j\,(j \in J \Rightarrow z_j \in A_j)$ AND $\forall j\,(j \in J \Rightarrow z_j \in B_j)$. Together with the notion of the product space this yields $z = (z_j)_{j\in J} \in \prod_{j\in J} A_j$ AND $z = (z_j)_{j\in J} \in \prod_{j\in J} B_j$, which by using the intersectioning idea produces the goal $z = (z_j)_{j\in J} \in \prod_{j\in J} A_j \cap \prod_{j\in J} B_j$. Since the element z was chosen arbitrarily out of the initial set the desired subset relation is established. ■

2 **Statement.** ***(Unions of Boxes)*** *In any product space unions of (two) boxes are inside a (minimal) box, which is the product set given by the union of the factor sets:*

$$\forall J \Big(J \text{ set } \Rightarrow \forall (X_j)_{j\in J}\Big(\forall j\,(j \in J \Rightarrow X_j \text{ set and} \,\forall (A_j,\, B_j \subset X_j)) \Rightarrow \prod_{j\in J} A_j \bigcup \prod_{j\in J} B_j \subset \prod_{j\in J}(A_j \bigcup B_j) \quad))$$

Proof Indication: Like the "$\subset$" proof above only substitute "$\cap$" by "$\cup$" and exchange "AND" for "OR". ■

Remark: The reverse containment cannot be shown in that generality, which means generally: $\prod_{j\in J} A_j \cup \prod_{j\in J} B_j \not\supset \prod_{j\in J}(A_j \cup B_j)$; the dual argument fails within the sentence "Together with the notion ..." because the element z cannot be identified as being in one of the product sets. As a simple counterexample for the non-containment consider the product space $\{0,1\} \times \{a,b\}$ (graphically); within that space the union of two boxes ( and ) $\{(0,a)\} \cup \{(1,b)\} = \{(0,a),(1,b)\}$ () clearly does not contain the single box $(\{0\} \cup \{1\}) \times (\{a\} \cup \{b\}) = \{(0,a),(0,b),(1,a),(1,b)\}$ ( represents the entire space). Another source of an example constitutes the diagram on the previous page, there the white space in the enclosing rectangle contains the points in question.

Remark: The relations above can be generalized easily to equations like

$$\bigcap_{k\in K}\prod_{j\in J} A_{j,k} = \prod_{j\in J}\bigcap_{k\in K} A_{j,k} \qquad \text{and} \qquad \bigcup_{k\in K}\prod_{j\in J} A_{j,k} \subset \prod_{j\in J}\bigcup_{k\in K} A_{j,k}$$

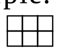
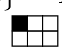
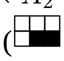
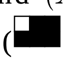
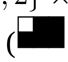
3 **Statement.** ***(Difference of Boxes)*** *The entire product space taken without a box contains a product set and is equal to a union of sets as described by the following formulas:*

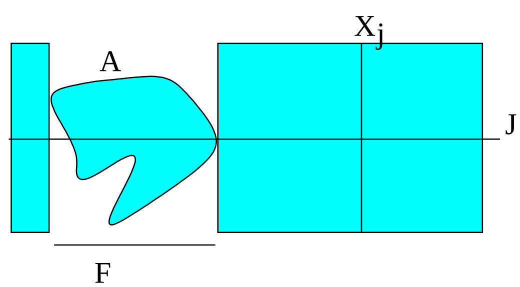
$$\forall J \Big(J \text{ set } \Rightarrow \forall (X_j)_{j\in J}\Big(\forall j\,(j \in J \Rightarrow X_j \text{ set and} \,\forall (A_j,\, B_j \subset X_j)) \Rightarrow$$
$$\prod_{j\in J}(X_j \backslash A_j) \subset \prod_{j\in J} X_j \backslash \prod_{j\in J} A_j = \bigcup_{j\in J}\Big(X_j \backslash A_j \times \prod_{k\in J\backslash\{j\}} X_k\Big)$$

Text (ISBN 3-00-004696-8) to be found at http://www.schmarsow.de

Proof Indication: “first $\subset$” If the set $\prod_{j\in J}(X_j\backslash A_j)$ is empty then the subset relation is true; if that set is not empty, then there exists an element $z\in\prod_{j\in J}X_j$ inside. This and the definition of the product space specifies the element into $z=(z_j)_{j\in J}$ with the factor elements $z_j\in X_j\backslash A_j$ or $z_j\notin A_j$ for all indices $j\in J$. By taking an arbitrary element z_{i_0}, there exists a factor element in $X_{i_0}\backslash A_{i_0}$, so the element $z=(z_j)_{j\in J}$ cannot be inside $\prod_{j\in J}A_j$ and thus is inside $\prod_{j\in J}X_j\backslash\prod_{j\in J}A_j$.
“second $\subset$” If the set $\prod_{j\in J}X_j\backslash\prod_{j\in J}A_j$ is empty the statement is true; otherwise there exists an element $w\in\prod_{j\in J}X_j$ and an index $j_0\in J$ so that the factor element $w_{j_0}\notin A_{j_0}$ is not inside the associated factor set. So this product element w is inside the product set $X_{j_0}\backslash A_{j_0}\times\prod_{k\in J\backslash\{j_0\}}X_k$ and thus within the union of all these sets taken over the index $j_0\in J$.
“second $\supset$” If the set $\bigcup_{j\in J}\left(X_j\backslash A_j\times\prod_{k\in J\backslash\{j\}}X_k\right)$ is empty, then the subset relation is true; if that set is not empty, then there must exist an index $j_\star\in J$ so that $X_{j_\star}\backslash A_{j_\star}\times\prod_{k\in J\backslash\{j_\star\}}X_k$ contains an element $s=(s_j)_{j\in J}$with $s_{j_\star}\notin A_{j_\star}$ and $s_j\in X_j$ for all other indices $j\in J\backslash\{j_\star\}$. This establishes the relation $s\in\prod_{j\in J}X_j\backslash\prod_{j\in J}A_j$, to be shown. ■

Remark: The relative simplicity of the argument shows that the relations proved above are merely reformulations or “obvious” inclusions.

Example: As an example consider the product space $\{0,1,2\}\times\{a,b\}=X_1\times X_2$ (graphically ); and sets $A_1=\{0\}$ and $A_2=\{b\}$ ($A_1\times A_2=\{(0,b)\}$ or ) then $X_1\backslash A_1\times X_2\backslash A_2=\{1,2\}\times\{a\}$, which are two elements (), and $(X_1\times X_2)\backslash(A_1\times A_2)=(\{0,1,2\}\times\{a,b\})\backslash\{(0,b)\}$ which are six elements except one (). Especially the union of factor-wise differences $(X_1\backslash A_1\times X_2)\cup(X_1\times X_2\backslash A_2)=(\{1,2\}\times\{a,b\})\cup(\{0,1,2\}\times\{a\})$ is explicitly given by the set $\{(0,a),(1,a),(2,a),(1,b),(2,b)\}$ (), this is just what has been predicted.



4 **Statement.** ***(Difference of Sets)*** *Consider the (index) set J and a family $(X_j)_{j\in J}$ of factor sets that make up the product set $X:=\prod_{j\in J}X_j$. For an arbitrary subset $F\subset J$ consider the product sets $X_F:=\prod_{f\in F}X_f$, $X_{J\backslash F}:=\prod_{j\in J\backslash F}X_j$ and use the isomorphism of sets $X\cong X_F\times X_{J\backslash F}$ for identification, then can be said:*

$$\forall A\left(A\subset X_F \text{ and } X_{J\backslash F}\neq\emptyset\Rightarrow A\times X_{J\backslash F}=X\backslash\left((X_F\backslash A)\times X_{J\backslash F}\right)\right)$$

Proof Indication: Use the notation: $x_F \in X_F$ and $x_{J\backslash F} \in X_{J\backslash F}$. If the set A is empty, the statement is true, the same is the case for $A = X_F$. (If the index sets J and F are equal then the product set $X_{J\backslash F}$ is a singleton, the set that contains the empty function.) So consider a non-empty set $A \times X_{J\backslash F}$ with $A \neq X_F$. The second condition, together with the prerequisite $X_{J\backslash F} \neq \emptyset$ makes also $X\backslash \left((X_F\backslash A) \times X_{J\backslash F}\right)$ nonempty.
Note the following equivalence for an element
$x = (x_F, x_{J\backslash F}) \in X_F\backslash A \times X_{J\backslash F} \Leftrightarrow x_F \notin A \,\text{and}\, \forall j\,(j \in J\backslash F \Rightarrow x_j \in X_j)$, giving negated
$x \in X\backslash \left(X_F\backslash A \times X_{J\backslash F}\right) \Leftrightarrow x_F \in A$ suffices to show the result. ∎

Generalized Direct Sums

The generalized direct sum is a special kind of subset of a product set. In most cases the generalized direct sum is not a box:

5 **Definition** ***(Generalized Direct Sum)*** *Inside a product space $\prod_{j\in J} X_j$ consider an element $a = (a_j)_{j\in J}$ and a family $(A_j)_{j\in J}$ of subsets ($A_j \subset X_j$). Then the set of all families which are equal to the family a except for finitely many factor elements*

$$\bigoplus_{j\in J}^{a} A_j := \left\{ (x_j)_{j\in J} \,\middle|\, \forall j\,(j \in J \Rightarrow x_j \in A_j) \text{ and } \exists F\,(F \text{ finite} \subset J \text{ and } \forall j\,(j \in J\backslash F \Rightarrow x_j = a_j)) \right\}$$

is called generalized direct sum with respect to the defining element a. *Or that set is called* a-direct sum of the given family of subsets. *The defining element is taken to be inside the generalized direct sum $a \in \bigoplus_{j\in J}^{a} A_j$, unless specified differently. (The idea for the definition has been taken from [TGI.28 §4.3 Prop. 8 Object D])*

If the index set J contains only finitely many elements, then the generalized direct sum is a box. (Generalized direct sums are generalizations of direct sums of vector spaces as defined in [AII.12 §1.6 first paragraph])

Topological Closures

A product set of topological spaces can be given a topology that is constructed from the topologies of its factor sets. Conceptually simple ones are the product topology $\mathcal{O}_\pi$ ([TGI.14 §2.3 Example III] or [WGT.53 Def. 8.3]) and the box topology $\mathcal{O}_{\text{box}}$ ([TGI.95 §4 Exercise 9)] or [WGT.53 the paragraph facing Def. 8.3]). The next statement is a generalization of well-known results [TGI.27 §4.3 Prop. 7 and TVSII.55 6.2.]:

6 **Statement.** ***(Product Closure of a Box)*** *Consider a (non-empty) (index) set J, which may be countable or use the Axiom of Choice*[1]*. Furthermore consider a family of topological spaces $((X_j, \mathcal{O}_j))_{j\in J}$ were each of them is non-empty, $\forall j\,(j \in J \Rightarrow X_j \neq \emptyset)$. On the product set $X = \prod_{j\in J} X_j$ of these spaces consider a topology $\mathcal{O}$ that is coarser than the*

[1] In this context the Axiom of Choice presents itself in the following form: $\forall j\,(j \in J \Rightarrow X_j \neq \emptyset) \Rightarrow \prod_{j\in J} X_j \neq \emptyset\ (J \neq \emptyset)$ Further information about that Axiom can be found in [WGT.9 1.17] and [WGT.52 preceding 8.2].

box topology but finer than the product topology $\mathcal{O}_\pi \subset \mathcal{O} \subset \mathcal{O}_{\text{box}}$ *. Then the closure of a box is the product of the closures of its factor sets:*

$$\mathrm{cl}_{\mathcal{O}}\left(\prod_{j\in J} A_j\right) = \prod_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j)$$

Proof Indication: "$\subset$" The product topology [TGI.13 §2.3 Prop.5 Expl. III] makes the projections[2] $\mathrm{pr}_k : X \to X_k$, $(x_j)_{j\in J} \mapsto x_k$ $\mathcal{O}_\pi - \mathcal{O}_k$-continuous. Since the product topology $\mathcal{O}_\pi \subset \mathcal{O}$ is coarser than the topology $\mathcal{O}$ on the product, the projections are also $\mathcal{O}$-$\mathcal{O}_k$ -continuous. This continuity is equivalent [TGI.9 §2.1 Th. 1 a) and b)] to the subset relation: $\mathrm{pr}_k\left(\mathrm{cl}_{\mathcal{O}}\left(\prod_{j\in J} A_j\right)\right) \subset \mathrm{cl}_{\mathcal{O}_k}\left(\mathrm{pr}_k\left(\prod_{j\in J} A_j\right)\right) = \mathrm{cl}_{\mathcal{O}_k}(A_k)$. Consider the J-product of the above (taking a product conserves subset relations) and remember that a subset in a product set is always inside the box built by its projections [E.R.22 §4.12 (49)], giving $\mathrm{cl}_{\mathcal{O}}\left(\prod_{j\in J} A_j\right) \subset \prod_{k\in J} \mathrm{cl}_{\mathcal{O}_k}(A_k)$.

"$\supset$" The statement is true for an empty box. In a non-empty box $\prod_{j\in J} A_j$ consider an element $b = (b_j)_{j\in J} \in \prod_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j)$ and open neighborhoods $U_j \in \mathcal{O}_j(b_j)$, forming an element $U := \prod_{j\in J} U_j \in \mathcal{B}_{\text{box}}(b)$ of the neighborhood base of the box topology. With the closure definition [TGI.7 §1.6 Def. 10] in a topological space $(X_j, \mathcal{O}_j)$ follows $U_j \cap A_j \neq \emptyset$. This non-emptiness, together with the denumerability of the index set J or with the Axiom of Choice can be used to deduce $U \cap \prod_{j\in J} A_j \neq \emptyset$.

By the choice of U_j any element U of the neighborhood base $\mathcal{B}_{\text{box}}(b)$ can be considered, the definition of a neighborhood base [TGI.4 §1.3 Def. 5] for the box topology validates the left side of the closure defining equivalence $\forall W\left(W \in \mathcal{O}_{\text{box}}(b) \Rightarrow W \cap \prod_{j\in J} A_j \neq \emptyset\right) \Leftrightarrow b \in \mathrm{cl}_{\mathcal{O}_{\text{box}}}\left(\prod_{j\in J} A_j\right)$. Since, as presupposed, the topology on the product space is coarser than the box topology and coarser topologies make bigger closures[3], the following inclusion relation is established: $\prod_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j) \subset \mathrm{cl}_{\mathcal{O}_{\text{box}}}\left(\prod_{j\in J} A_j\right) \subset \mathrm{cl}_{\mathcal{O}}\left(\prod_{j\in J} A_j\right)$. ■

7 **STATEMENT.** ***(Product Closure of a Generalized Direct Sum)*** *Consider a (non-empty) (index) set J, which may be countable or use the Axiom of Choice. Furthermore consider a family of topological spaces $((X_j, \mathcal{O}_j))_{j\in J}$, with a family of non-empty subsets $\forall j\,(j \in J \Rightarrow \exists a_j\,(a_j \in A_j \subset X_j))$. On the product set $X = \prod_{j\in J} X_j$ of the topological spaces consider the product topology $\mathcal{O}_\pi$. Then the closure of a generalized direct sum is the product of the closures of its factor sets:*

$$\mathrm{cl}_{\mathcal{O}_\pi}\left(\bigoplus_{j\in J}^{a} A_j\right) = \prod_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j)$$

Proof Indication: "$\subset$" A generalized sum, as the one above, always is inside the box of its factor sets. Taking the closure with respect to the product topology conserves

[2] In this text the notion *projection* follows the definition found in the context of the product spaces. — Maps $\mathrm{p} : X \to X$ that are surjective $\mathrm{p}(X) = X$ and idempotent $\mathrm{p} \circ \mathrm{p} = \mathrm{p}$ automorphisms often are also called "projections", here they are called *(idempotent) projections*. (The circle $\circ$ indicates the composition of mappings.)

[3] For a given set X with topologies $\mathcal{O}, \mathcal{O}_{\text{finer}}$ and a subset $A \subset X$ follows $\mathcal{O} \subset \mathcal{O}_{\text{finer}} \Rightarrow \mathrm{cl}_{\mathcal{O}_{\text{finer}}}(A) \subset \mathrm{cl}_{\mathcal{O}}(A)$.

the containment relation, giving $\mathrm{cl}_{\mathcal{O}_\pi}\left(\bigoplus^a_{j\in J} A_j\right) \subset \mathrm{cl}_{\mathcal{O}_\pi}\left(\prod_{j\in J} A_j\right)$. The rest follows by applying the first part of the previous statement.

"$\supset$" Since the factor sets A_j are non-empty, there exist elements $x \in \prod_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j) \subset \mathrm{cl}_{\mathcal{O}_\pi}\left(\prod_{j\in J} A_j\right)$, where the containment relation has been proved above.

Consider an x-neighborhood $U \in \mathcal{B}_{\mathcal{O}_\pi}(x)$ of the product topology: First, by the definition of the topological closure and the result above exists an element $y = (y_j)_{j\in J} \in U \cap \prod_{j\in J} A_j$. Secondly, the neighborhood being in the base of the product topology, can be rewritten as a product set $U = \prod_{j\in J} U_j$, with a finite subset $F \subset J$ of the index set, and factor sets $U_j = \begin{cases} V_j \in \mathcal{O}_j(x) & \text{if } j \in F, \\ X_j & \text{if } j \in J\backslash F \end{cases}$ as specified. With the first observation follows $y_j \in U_j \cap A_j$ (*) for all indices. Since, as presupposed, $a_j \in A_j$ and as noticed in the rewriting of the x-neighborhood, $\forall j\,(j \in J\backslash F \Rightarrow U_j = X_j)$ follows $a_j \in U_j \cap A_j$ (**) for indices $j \in J\backslash F$. Now define an element $z := (z_j)_{j\in J}$ with $z_i = \begin{cases} y_j & \text{if } j \in F, \\ a_j & \text{if } j \in J\backslash F. \end{cases}$ By (*), (**), the definition of U and the definition of the generalized sum follows $z \in U \cap \bigoplus^a_{j\in J} A_j$, making this intersection non-empty. The arbitrary choice of U makes that equivalent to $x \in \mathrm{cl}_{\mathcal{O}_\pi}\left(\bigoplus^a_{j\in J} A_j\right)$ by the definition of a topological closure. ■

8 **STATEMENT.** ***(Box Closure of a Generalized Direct Sum)*** *Consider a non-empty (index) set J. Furthermore consider a family of T_1-topological[4] spaces $((X_j, \mathcal{O}_j))_{j\in J}$, with a family of non-empty subsets $\forall j\,(j \in J \Rightarrow \exists a_j\,(a_j \in A_j \subset X_j))$. On the product set $X = \prod_{j\in J} X_j$ of the topological spaces consider the box topology $\mathcal{O}_{\text{box}}$. Then a generalized direct sum is topologically closed:*

$$\mathrm{cl}_{\mathcal{O}_{\text{box}}}\left(\bigoplus^a_{j\in J} A_j\right) = \bigoplus_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j)$$

Idea from [EVTI.30 §1 Exercise 14)]

Proof Indication: If the index set J is finite then the statement becomes a special case of a statement before, so the (second part) of the proof uses an index set J with infinitely many indices. "$\supset$" Consider an element $x = (x_j)_{j\in J} \in \bigoplus^a_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j)$, by the definition of the generalized direct sum there exists a finite subset $F \subset J$ of the index set for which the coordinates of the element x may differ from those of the element a: $\forall j\,(j \in J\backslash F \Rightarrow x_j = a_j)$ (*).

Next consider a box-neighborhood $U = \prod_{j\in J} U_j \in \mathcal{B}_{\mathcal{O}_{\text{box}}}(x)$ of the element x; by the definition of the closure inside the generalized sum above, follows for all coordinates f of the finite subset $F \subset J$: $\exists x'_f\left(x'_f \in U_f \cap A_f\right)$ (**).

Now define an element $s = (s_j)_{j\in J}$ by $s_j := \begin{cases} x'_j & \text{if } j \in F, \\ x_j & \text{if } j \in J\backslash F. \end{cases}$ From (*) and (**) follows $s \in U$ and $s \in \bigoplus^a_{j\in J} A_j$, giving $U \cap \bigoplus^a_{j\in J} A_j \neq \emptyset$. Since the box-neighborhood was not

[4] T_1-topolgical means that $\forall x, y\,(x, y \in X \Rightarrow \exists U\,(U \in \mathcal{O}(x) \text{ and } y \notin U))$, this is somewhat more general than the Hausdorff property where every pair of non-equal elements has a corresponding pair of non-intersecting open neighborhoods.

restricted in any way the closure definition gives $x \in \mathrm{cl}_{\mathcal{O}_{\mathrm{box}}}\left(\bigoplus^a_{j\in J} A_j\right)$.
"$\subset$" Reformulate the inclusion relation element-wise and use the relation $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ to get the to be shown: $\forall x \left(x \notin \bigoplus_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j) \Rightarrow x \notin \mathrm{cl}_{\mathcal{O}_{\mathrm{box}}}\left(\bigoplus^a_{j\in J} A_j\right)\right)$ Using the definition of the generalized direct sum, rewrite the presupposition $x \notin \bigoplus_{j\in J} \mathrm{cl}_{\mathcal{O}_j}(A_j)$ giving
$\exists j_0 \left(j_0 \in J \,\mathrm{and}\, x_{j_0} \notin \mathrm{cl}_{\mathcal{O}_{j_0}}(A_{j_0})\right)$ or $\forall F\, (F \subset J$ and F finite $\Rightarrow \exists j_F\, (j_F \in J\backslash F \,\mathrm{and}\, x_{j_F} \neq a_{j_F}))$.
So there are two cases to be handled[5]:
First, define a set $U = \prod_{j\in J} U_j$ by $U_j = \begin{cases} X_{j_0}\backslash \mathrm{cl}_{\mathcal{O}_{j_0}}(A_{j_0}) & \text{if } j = j_0, \\ X_j & \text{if } j \in J\backslash\{j_0\}. \end{cases}$ The presupposed and the definition give $U \in \mathcal{B}_{\mathcal{O}_{\mathrm{box}}}(x)$ and $U \cap \bigoplus^a_{j\in J} A_j = \emptyset$, thus establishing the to be shown $x \notin \mathrm{cl}_{\mathcal{O}_{\mathrm{box}}}\left(\bigoplus^a_{j\in J} A_j\right)$ for the first case.
In the second case consider the set of all those indices in which the families x and a differ: $I_x := \{j \,|\, j \in J \,\mathrm{and}\, x_j \neq a_j\}$ By the presupposition, and the fact that the set of all finite subsets is countable, an induction process yields: I_x is a set with infinitely many elements. The T_1-property of the factor topologies permits to identify open neighborhoods of the following kind: $\forall j\, (j \in I_x \Rightarrow X_j\backslash\{a_j\} \in \mathcal{O}_j(x_j))$. Now define a set $U := \prod_{j\in J} U_j$ by $U_j := \begin{cases} X_j\backslash\{a_j\} & \text{if } j \in I_x, \\ X_j & \text{if } j \in J\backslash I_x. \end{cases}$ The presupposed and the definition give $U \in \mathcal{B}_{\mathcal{O}_{\mathrm{box}}}(x)$.
Since all elements $u \in U$ have infinitely many factor elements $u_j \neq a_j$, especially those with indices in I_x, they are not elements of the direct sum set $\bigoplus^a_{j\in J} A_j$, thus follows $U \cap \bigoplus^a_{j\in J} A_j = \emptyset$, establishing the to be shown $x \notin \mathrm{cl}_{\mathcal{O}_{\mathrm{box}}}\left(\bigoplus^a_{j\in J} A_j\right)$ for the second case.
■

Some Product Spaces

A product set of groups [AI.28 §4.1 Def. 1] (or vector spaces [AII.3 §1.1 Def. 2]) can be given the structure of a group (or a vector space). This is done by defining the operations using the operations of its factor spaces component-wise as described in [AI.43 §4.8 Def. 12] (or [AII.10 §1.5 first paragraph]). For families of topological groups (or vector spaces) the algebraic product structures can be supplemented by topological ones; as presented in the following sections, there the different properties concerning the summability of elements are investigated.
Generally [TGIII.37 §5.1 Remark 3)], *families* $(s_i)_{j\in J}$ *of elements that are summable*, can be introduced in a commutative monoid $(M, +)$ [AI.12 §2.1 Def. 1] which has a Hausdorff topology $\mathcal{O}$ with a neighborhood filter $\mathcal{U}_{\mathcal{O}}(s)$ of an element $s \in M$, $\forall j\, (j \in J \Rightarrow s_j \in M)$:

$$\sum_{j\in J} s_j := s \,\middle|\, \forall U \left(U \in \mathcal{U}_{\mathcal{O}}(s) \Rightarrow \exists F_U \left(F_U \text{ finite } \subset J \,\mathrm{and}\, \forall F \left(F_U \subset F \text{ finite } \subset J \Rightarrow \sum_{j\in F} s_j \in U\right)\right)\right) .$$

Furthermore, the *support of a family* $(y_x)_{x\in X} \in \prod_{x\in X} Y_x$, where $(Y_x)_{x\in X}$ is a given family of usually commutative monoids with neutral elements which are all denoted zero 0, is

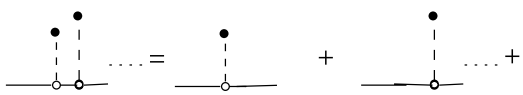
[5] The general strategy becomes: For a given element x find a box-neighborhood U that does not intersect the generalized direct sum $\bigoplus^a_{j\in J} A_j$.

defined to be the set $\operatorname{supp}\left((y_x)_{x\in X}\right) = \{x \,|\, x \in X \text{ and } y_x \neq 0\}$ of indices that have a non-zero value.

Product Groups and Product Topology

9 **NOTE.** ***(Topological Product Groups (Product Topology))*** *If the factor spaces in the family $((E_j, +, \mathcal{O}_j))_{j\in J}$ are topological groups, then the product space $E := \prod_{j\in J} E_j$ is a topological group with respect to the group operation "+" defined component-wise, the neutral element "0" being the family of all neutral elements, the inverse being the family of factor-wise inverses and the product topology $\mathcal{O}_\pi$. The topological space $(E, \mathcal{O}_\pi)$ is Hausdorff if the factor spaces are Hausdorff. [TGIII.17 §2.9 second sentence]*

For an index $k \in J$ consider the map $\mathrm{j}_k : E_k \to \prod_{j\in J} E_j$ given by $x_k \mapsto (x_j)_{j\in J} := \begin{cases} x_k & \text{if } j = k, \\ 0_j & \text{if } j \in J\backslash\{k\}, \end{cases}$ giving an image $E_{(k)} := \left\{ (x_j)_{j\in J} \middle| \forall j\,(j \in J\backslash\{k\} \Rightarrow x_j = 0) \right\}$. The map[6] is injective and thus an isomorphism onto its image. An image element of that map is called *singly supported* because, except for maximally one coordinate, all other coordinates are zero.

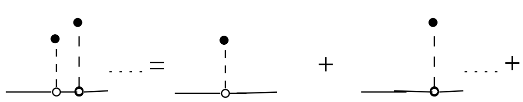


up to infinitely many

10 **NOTE.** ***(Summability of Singly Supported Elements (Product Topology))*** *Consider the injections $\mathrm{j}_k : E_k \to \prod_{j\in J} E_j$ $(k \in J)$ of the factor sets into the product set and let the topologies of the factor spaces be Hausdorff. Then any element $x = (x_j)_{j\in J} \in \prod_{j\in J} E_j$ can be broken down to a $\mathcal{O}_\pi$-summable family $(\mathrm{j}_k(x_k))_{k\in J}$ with $x = \sum_{k\in J} \mathrm{j}_k(x_k)$ where the sum is taken with respect to the product topology. [special case of TGIII.41 §5.4 Prop. 4]*

Product Groups and Box Topology

11 **NOTE.** ***(Topological Product Groups (Box Topology))*** *If the factor spaces in the family $((E_j, +, \mathcal{O}_j))_{j\in J}$ are topological groups, then the product space $E := \prod_{j\in J} E_j$ is a topological group with respect to the group operation "+" defined component-wise, the neutral element "0" being the family of all neutral elements, the inverse being the family of factor-wise inverses and the box topology $\mathcal{O}_{\text{box}}$. The topological space $(E, \mathcal{O}_{\text{box}})$ is Hausdorff if the factor spaces are Hausdorff. [TGIII.70 §2 Exercise 23)]*



finitely many

[6] The projection $\mathrm{pr}_k : \prod_{j\in J} E_j \to E_k$, given by $(x_j)_{j\in J} \mapsto x_k$, and the subset $E_{(k)}$ make the restriction $\mathrm{pr}_k\big|_{E_{(k)}}$ an isomorphism of groups. Its inverse is the injection $\mathrm{j}_k\big|^{E_{(k)}}$ corestricted to its image.

12 **Statement.** ***(Summability of Singly Supported Elements (Box Topology))*** *Consider the injections* $\mathrm{j}_k : E_k \to \prod_{j\in J} E_j$ *($k \in J$) of the factor sets into the product set and let the topologies of the factor spaces be Hausdorff. Then exactly those elements* $x = (x_j)_{j\in J} \in \prod_{j\in J} E_j$ *can be broken down to a* $\mathcal{O}_{\text{box}}$*-summable family* $(\mathrm{j}_k(x_k))_{k\in J}$ *with sums* $x = \sum_{k\in J} \mathrm{j}_k(x_k)$*, that are the elements with finitely many non-zero factor elements:*

$$((\mathrm{j}_k(x_k))_{k\in J} \text{ is } \mathcal{O}_{\text{box}} - \text{summable to } s \Leftrightarrow \text{supp}(x) \text{ is finite}) \Rightarrow x = \sum_{k\in J} \mathrm{j}_k(x_k)$$

Proof Indication: "⇐" Any sum of finitely many summands is summable in any topology. "⇒" For every index $s \in \text{supp}(x)$ follows $x_s \neq 0_s$ and by the Hausdorff property exists an open neighborhood $U'_s \in \mathcal{O}(x_s)$ with $0 \notin U'_s$ (*). With this define a box-neighborhood $U := \prod_{j\in J} U_j$ of the element x by $U_j := \begin{cases} U'_j & \text{if } j \in \text{supp}(x), \\ E_j & \text{if } j \in J\backslash\text{supp}(x). \end{cases}$

Since, as presupposed, the family $(\mathrm{j}_k(x_k))_{k\in J}$ is $\mathcal{O}_{\text{box}}$-summable, for the x-neighborhood U exists a finite subset $F \subset J$ of the index set and $\sum_{f\in F} \mathrm{j}_f(x_f) \in U$. Now reconsider that finite sum, it can be written as a sum over an element $y := (y_j)_{j\in J}$ with $y_j := \begin{cases} x_j & \text{if } j \in F, \\ 0_j & \text{if } j \in J\backslash F, \end{cases}$ and $\sum_{j\in J} \mathrm{j}_j(y_j) = \sum_{f\in F} \mathrm{j}_f(x_f) \in U$. For indices $k \in J\backslash F$ follows $0 = y_k \in U_k$, and by (*) follows $k \notin \text{supp}(x)$, thus $\text{supp}(x) \subset F$ and therefore $\text{supp}(x)$ is finite.

Finally show that the sum $s = \sum_{k\in J} \mathrm{j}_k(x_k)$, where $s = (s_j)_{j\in J}$, is equal to the element x: Since the sum is made up of finitely many summands, using the homomorphy of the projection and a Kronecker symbol for groups ($\delta_{j,k}x_k := \begin{cases} 0 & \text{if } k \neq j \\ x_k & \text{if } k = j \end{cases}$) the following can be shown:

$s_j = \text{pr}_j(s) = \sum_{k\in J} \text{pr}_j(\mathrm{j}_k(x_k)) = \sum_{k\in J} \delta_{j,k}x_k = x_j$ ■

Those factor sets of the neighborhood U, with indices inside the support, never contain zeros! Using this neighborhood U, the summability gives a finite index set wherein a potential approximation of summands of the sum are defined. But exactly that finite sum being inside the neighborhood U forces the support of the original summands into the finite set!

Product Vector Spaces and Box Topology

Direct sum space

The continuity of a map can be formulated in various ways. The common approach includes a formulation with respect to a point of reference (locally) [TGI.8 §2.1 Def. 1] or without a point of reference (globally) [TGI.9 §2.1 Def. 2]. This notion of a continuous function can be specialized for homomorphisms between groups (or vector spaces ...) with a topology. Another special case arises when the group operations are asked to be continuous [TGIII §1.1 Def. 1]: In these cases the topology and the group operation can be tested locally for the continuity of the group operation or — a given filter can be tested for whether it can be used to construct a topology that makes the operation continuous [TGIII.3 §1.2 Prop. 1]. The same can be seen with rings [TGIII.48 §6.3 Def. 2] and vector spaces/modules [TGIII.52 §6.6 Def. 3 and $(\mathrm{MV_I} - \mathrm{MV_{III}})$]:

13 **Note.** ***(Continuity of Scalar Multiplication)*** *Consider a vector space $(E, +, \mathbb{K})$ and consider a filter*[7] *$\mathcal{U}_E$ on the set E. The scalar multiplication being continuous is equivalent to three valid conditions:*

(preabsorbtion) $\forall U, x\, (U \in \mathcal{U}_E \text{ and } x \in E \Rightarrow \exists \Gamma\, (\Gamma \in \mathcal{U}_{\mathbb{K}}(0) \text{ and } \Gamma \cdot x \subset U))$

(prebalance) $\forall U\, (U \in \mathcal{U}_E \Rightarrow \exists \Gamma, V\, (\Gamma \in \mathcal{U}_{\mathbb{K}}(0) \text{ and } V \in \mathcal{U}_E \text{ and } \Gamma \cdot V \subset U))$

(pre) $\forall U, s\, (U \in \mathcal{U}_E \text{ and } s \in \mathbb{K} \Rightarrow \exists V\, (V \in \mathcal{U}_E \text{ and } s \cdot V \subset U))$

(Each element of the filter must automatically contain the null vector.)

14 **Statement.** ***(Direct Sum with Restricted Box Topology remains a Vector Space)*** *Consider a family $((E_j, +, \mathbb{K}, \mathcal{O}_j))_{j\in J}$ of topological vector spaces. The direct sum space $\bigoplus_{j\in J} E_j$ [AII.12 §1.6 first paragraph] is a vector space with respect to operations which are defined component-wise and where the zero is the family of zeros. Further consider the box topology $\mathcal{O}_{\text{box}}$ on the product space $E = \prod_{j\in J} E_j$, this topology can be induced $\mathcal{O}^{\bigoplus}_{\text{box}} := E \cap \mathcal{O}_{\text{box}}$ onto the direct sum space. If the ground field $\mathbb{K}$ has an absolute value then the direct sum space is a topological vector space $\left(\bigoplus_{j\in J} E_j, +, \mathbb{K}, \mathcal{O}^{\bigoplus}_{\text{box}}\right)$ with respect to the induced box topology. [EVTI.30 §1 Exercise 14) and TVSII.55 6.2]*

Proof Indication: The filter $\mathcal{U}_{E_\oplus} := \left\langle \mathcal{B}^{\oplus}_{\text{box}}(0) \right\rangle_{\text{filter}}$ is taken to be generated by the 0-filter base of the induced box topology; and write $E_{\bigoplus} := \bigoplus_{j\in J} E_j$.
(pre) Consider a field element $s \in \mathbb{K}$ and an open zero neighborhood $U = \prod_{j\in J} U_j \cap E_{\bigoplus} \in \mathcal{B}^{\oplus}_{\text{box}}(0)$. As presupposed, each factor space $(E_j, +, \mathbb{K}, \mathcal{O}_j)$ is a topological vector space, each of them validates the Condition (pre), so that there exists $V_j \in \mathcal{U}_{E_j}$ and $s \cdot V_j \subset U_j$ from which follows immediately $s \cdot \prod_{j\in J} V_j \subset \prod_{j\in J} s \cdot V_j \subset \prod_{j\in J} U_j$ (*). Now define a set $V := \prod_{j\in J} V_j \cap E_\oplus$ which is inside $\mathcal{B}^{\oplus}_{\text{box}}(0) \subset \mathcal{U}_{E_\oplus}$ and that satisfies $s \cdot V \subset U$, which can be seen by restricting Observation (*) to $E_\oplus$.
(preabsorbtion) Consider an element $x \in E_\oplus$, which means that $x = (x_j)_{j\in J}$ has finitely many non-zero elements or a finite support, and consider an open zero neighborhood $U = \prod_{j\in J} U_j \cap E_{\bigoplus} \in \mathcal{B}^{\oplus}_{\text{box}}(0)$. As presupposed, each factor space $(E_j, +, \mathbb{K}, \mathcal{O}_j)$ is a topological vector space, each of them validates the Condition (preabsorbtion), so that for each $j \in J$ there exists a zero-neighborhood $\Gamma_j \in \mathcal{U}_{\mathbb{K}}(0)$ of the field, so that $\Gamma_j \cdot x_j \subset U_j$. Since

[7] A filter is a nonempty set of sets containing the intersections of finitely many of its elements and containing all supersets of its elements. [TGI.36 §6.1 Def. 1]

$\mathrm{supp}(x)$ is finite, the intersection $\Gamma_x := \cap_{j\in\mathrm{supp}(x)}\Gamma_j \subset \mathbb{K}$ remains a zero-neighborhood in $\mathcal{U}_{\mathbb{K}}(0)$. From this follows:
$\Gamma_x \cdot x = \prod_{j\in J}\Gamma_x \cdot x_j \subset \prod_{j\in J}\Gamma_j \cdot x_j \subset F := \prod_{j\in J} F_j,\ F_j := \begin{cases} U_j & \text{if } j \in \mathrm{supp}(x), \\ \{0_j\} & \text{if } j \in J\backslash\mathrm{supp}(x). \end{cases}$ This inclusion relation and $F \cap E_\oplus = F$ yield $\Gamma_x \cdot x \subset F \subset \prod_{j\in J} U_j \cap E_\oplus = U$.
(prebalance) Consider an open zero-neighborhood $U = \prod_{j\in J} U_j \cap E_{\bigoplus} \in \mathcal{B}^{\oplus}_{\mathrm{box}}(0)$. As presupposed, each factor space $(E_j, +, \mathbb{K}, \mathcal{O}_j)$ is a topological vector space, each of them validates the Condition (prebalance), so that there exist $V'_j \in \mathcal{U}_{E_j}$, $\Gamma_j \in \mathcal{U}_{\mathbb{K}}(0)$ and $\Gamma_j \cdot V'_j \subset U_j$ (***) (since $\mathcal{U}_{E_j} = \langle\mathcal{O}_j(0_j)\rangle_{\mathrm{filter}}$ substitute V'_j by $V_j \in \mathcal{O}_j(0_j)$ for later use). The absolute value of the field allows for a disk $\mathrm{U}_{\epsilon_j}(0) \subset \Gamma_j$, for all $j \in J$ with $0 < \epsilon_j \in \mathbb{R}$. And together with $\mathrm{U}_1(0)\,\mathrm{U}_\epsilon(0) \subset \mathrm{U}_\epsilon(0)$ follows $\mathrm{U}_1(0)\,\mathrm{U}_{\epsilon_j}(0)\cdot V_j \subset \mathrm{U}_{\epsilon_j}(0)\cdot V_j \subset \Gamma_j \cdot V_j \subset U_j$. This yields $\mathrm{U}_1(0)\prod_{j\in J}\mathrm{U}_{\epsilon_j}(0)\cdot V_j \subset \prod_{j\in J}\mathrm{U}_1(0)\,\mathrm{U}_{\epsilon_j}(0)\cdot V_j \subset \prod_{j\in J} U_j$ and with $\Gamma := \mathrm{U}_1(0)$ and $V := \prod_{j\in J}\mathrm{U}_{\epsilon_j}(0)\cdot V_j \cap E_\oplus$ this can be rewritten as $\Gamma \cdot V \subset U$, part of the to be shown. But it remains to verify that $V \in \mathcal{U}_{E_\oplus} = \left\langle \mathcal{B}^{\oplus}_{\mathrm{box}}(0)\right\rangle_{\mathrm{filter}}$ by showing that for all $j \in J$ the factor sets $\mathrm{U}_{\epsilon_j}(0)\cdot V_j \in \mathcal{U}_{E_j} = \langle\mathcal{O}_j(0_j)\rangle_{\mathrm{filter}}$ are zero-neighborhoods. By (***) the sets $V_j \in \mathcal{O}_j(0_j)$ are open and by [EVTI.3 §1.1 Prop. 1] which states that the mapping $x \mapsto s\cdot x : E_j \to E_j$ is a homeomorphism for $s \neq 0$, follows $s \cdot V_j \in \mathcal{O}_j$ for $s \neq 0$. And since the factor sets $\mathrm{U}_{\epsilon_j}(0)\cdot V_j = \cup_{s\in\mathrm{U}_{\epsilon_j}(0)\backslash\{0\}} s\cdot V_j$ can be represented as a union of open sets the factor sets are open. ■

Remarks:

(preabsorbtion) This part of the proof relies on the structure of the direct sum.

(prebalance) Here the absolute value of the ground field $\mathbb{K}$ furnishes an essential part of the proof. The splitting of the disk into a unit disk (a way to extract a common scalar factor set) and a rest $\mathrm{U}_1(0)\,\mathrm{U}_\epsilon(0) \subset \mathrm{U}_\epsilon(0)$ going into the V definition, finally makes it possible to show that $V \in \mathcal{U}_{E_\oplus}$ is in the filter.

Linear Sums in Hausdorff Topological Vector Spaces

In a topological vector space the notion of a linear combination [AII.3 §1.1 Def. 3] can be extended by considering sums of more than finitely many vectors. The result of such a summation is called linear sum. The same name is used for the ensuing equivalent of a linear span of a set of vectors.

15 **Definition** ***(A-Summability- and Linear Sum of a Family of Scalars)*** *Consider a Hausdorff topological vector space $(E, +, \mathbb{K}, \mathcal{O})$ over a Hausdorff topological field $\mathbb{K}$. A family of scalars $(x_a)_{a\in A} \in \mathbb{K}^A$, which is indexed by a subset $A \subset E$ of vectors is called* an (A-)summable family *if and only if the family $(x_a \cdot a)_{a\in A}$ is $\mathcal{O}$-summable in E.*
An element $s \in E$ is called linear sum of the family *$(x_a)_{a\in A} \in \mathbb{K}^A$ if and only if the family is A-summable to s. This limit element s is also written $\sum_{a\in A} x_a \cdot a$.*

Remark: Why not use an arbitrary index set that indexes also a family of vectors, resulting in terms like $\sum_{j\in J} x_j \cdot a_j$? This would be in accordance with the definition of a linear combination. This is more a question of notation, sums like $\sum_{j\in J} x_j \cdot a$ remain possible when considering a set $A = \{x_j \cdot a\}$ with different values of x_j. But the conventional notation would add unnecessary complexity to the following considerations and later applications.

16 **Definition** ***(Linear Sum of a Set)*** *As before consider a Hausdorff topological vector space $(E, +, \mathbb{K}, \mathcal{O})$ over a Hausdorff topological field $\mathbb{K}$. And consider a subset $A \subset E$ of vectors. The set S of all vectors which are linear sums of A-indexed families of scalars is called a* linear sum of the set A *in E, written:*

$$\operatorname{sum}_{\mathcal{O}}(A) := S = \left\{ s \,\middle|\, s \in E \text{ and } \exists (x_a)_{a\in A} \in \mathbb{K}^A \text{ and } (x_a \cdot a)_{a\in A}\ \mathcal{O} - \text{summable to } s \right\} \subset E$$

The set A is called the generating set of the linear sum *$\operatorname{sum}_{\mathcal{O}}(A)$, this is sometimes written simply $\operatorname{sum}(A)$. The set*

$$\$_A := \left\{ (x_a)_{a\in A} \,\middle|\, (x_a)_{a\in A} \in \mathbb{K}^A \text{ and } (x_a \cdot a)_{a\in A}\ \mathcal{O} - \text{summable} \right\} \subset \mathbb{K}^A$$

of all those families of scalars $(x_a)_{a\in A} \in \mathbb{K}^A$, which are A-summable is also called the pre-coordinate space of the linear sum.

Structures of Linear Sums and Relation to Linear Span

Of course all linear combinations of the vectors in the set A are contained in the linear sum: $\operatorname{span}(A) \subset \operatorname{sum}_{\mathcal{O}}(A)$. Different elements in the pre-coordinate space may sum the same element in the vector space.

17 **Statement.** ***(Pre-Coordinate Linear Map)*** *With the presuppositions given by the previous definitions follows:*

A linear sum $\text{sum}_{\mathcal{O}}(A) \leq E$ *is a topological vector subspace,*

the pre-coordinate space $\$_A \subset \mathbb{K}^A$ *is a vector space with respect to the componentwise operations and*

there exists a linear map $\varphi_A : \$_A \to E, \text{im}(\varphi_A) = \text{sum}_{\mathcal{O}}(A)$ *given by* $(x_a)_{a\in A} \mapsto \sum_{a\in A} x_a \cdot a$.

Proof Indication: A set S is a subspace of a vector space if S is non-empty, (i) $S+S \subset S$ and (ii) $\mathbb{K} \cdot S \subset S$ are valid [AII.6 first sentence of Section 3 and AI.31 §4.3 Def. 4]. For any set $A \subset E$, especially if $A = \emptyset$, follows $(0)_{j\in A} \in \$_A$ thus $\$_A \neq \emptyset$ (Also the linear sum $\text{sum}_{\mathcal{O}}(A)$ is always non-empty.). To show (i), consider two A-summable families $(x_a)_{a\in A}, (y_a)_{a\in A} \in \$_A$, elements of the pre-coordinate space (with $x = \sum_{a\in A} x_a \cdot a$ and $y = \sum_{a\in A} y_a \cdot a$ in E), so that by [TGIII.42 §5.5 Prop. 6(6)] (using the continuity of the addition in $(E, +, \mathcal{O})$), the family $(x_a \cdot a + y_a \cdot a)_{a\in A}$ is A-summable (to $x+y = \sum_{a\in A}(x_a+y_a)\cdot a$). To show (ii), consider a scalar $t \in \mathbb{K}$ and a family $(x_a)_{a\in A} \in \$_A$, an element of the pre-coordinate space. Then the continuity of the scalar multiplication in $(E, +, \mathbb{K}, \mathcal{O})$ yields $(t \cdot x_a)_{a\in A} \in \$_A$ (and $t \cdot x = \sum_{a\in A} t\, x_a \cdot a$). Thus the sets $\$_A$ and $\text{sum}_{\mathcal{O}}(A)$ are vector spaces, with respect to the operations used above. Considered together with the induced topology $\mathcal{O} \cap \text{sum}_{\mathcal{O}}(A)$, the linear sum is also a topological vector space [TGIII.53 §6.6 Second sentence past Remark]. The definedness of the map φ_A and the image relation $\text{im}(\varphi_A) = \text{sum}_{\mathcal{O}}(A)$ follow directly from the definitions of the notions. Furthermore, φ_A can be identified as linear, $\varphi_A((x_a)_{a\in A} + (y_a)_{a\in A}) = \varphi_A((x_a)_{a\in A}) + \varphi_A((y_a)_{a\in A})$ and $t \cdot \varphi_A((x_a)_{a\in A}) = \varphi_A(t \cdot (x_a)_{a\in A})$, by using the relations previously shown. ■

Note that the proof relies on the continuity of the vector addition and the scalar multiplication of the original (topological) vector space E.

Further properties of linear sums in topological vector spaces are given in the following statement:

18 **Statement.** ***(Subset Relations between Linear Sums)*** *The linear sums of subsets* $A, B \subset E$ *in a Hausdorff topological vector space* $(E, +, \mathcal{O}, \mathbb{K})$ *satisfy:*

$$(\subset) \quad A \subset B \Rightarrow \text{sum}(A) \subset \text{sum}(B)$$

$$(+.1) \qquad \text{sum}(A) + \text{sum}(B) \subset \text{sum}(A \cup B)$$

$$(+.2) \quad (E, +, \mathcal{O}) \text{ is complete} \Rightarrow \quad \text{sum}(A \cup B) \subset \text{sum}(A) + \text{sum}(B)$$

Proof Indication: $(\subset)$: Consider an element $x \in \text{sum}(A)$ which is a linear sum of a family $(x_a)_{a\in A} \in \$_A$ of scalars. Using the subset relation $A \subset B$ define a B-indexed family by $\overline{x}_b := \begin{cases} x_b & \text{if } b \in A, \\ 0 & \text{if } b \in B \backslash A. \end{cases}$ for $b \in B$. The extension by zeros makes A-summability to B-summability. Therefore follows $x = \varphi_A((x_a)_{a\in A}) = \varphi_B((\overline{x}_b)_{b\in B}) \in \text{sum}(B)$. —

(+.1): From ($\subset$) follow the inclusions $\mathrm{sum}(A), \mathrm{sum}(B) \subset \mathrm{sum}(A \cup B)$ and, as previously stated, all three sets are vector subspaces in E, yielding (+.1). — (+.2): Consider an element $x \in \mathrm{sum}(A \cup B)$ which is a linear sum of a family $(x_j)_{j \in A \cup B} \in \mathcal{S}_{A \cup B}$ of scalars. In the non-trivial case the sets A and B are non-empty, then the set $\{A, (A \cup B) \backslash A\}$ partitions the union $A \cup B$ and thus the inclusion $(A \cup B) \backslash A \subset B$ (*) is valid. By the completeness of the topological group $(E, +, \mathcal{O})$ and [TGIII.39 §5.3 Prop. 2] (subfamilies of summable families remain summable) follows that the subfamilies $(x_j \cdot j)_{j \in A}$ and $(x_j \cdot j)_{j \in (A \cup B) \backslash A}$ remain $\mathcal{O}$-summable. Moreover, [TGIII §5.3 Th. 2] a complete topological group allows the reassociation $x = \sum_{j \in A} x_j \cdot j + \sum_{j \in (A \cup B) \backslash A} x_j \cdot j$; with the first summand in $\mathrm{sum}(A)$, the second, using (*) and ($\subset$), in $\mathrm{sum}(B)$. ■

Remark: If the sets $\mathcal{S}_A$ and $\mathcal{S}_B$ are considered as injected (extended by zeros) into $\mathcal{S}_{A \cup B}$, then the above can be restated as $A \subset B \Rightarrow \mathcal{S}_A \subset \mathcal{S}_B$, $\mathcal{S}_A + \mathcal{S}_B \subset \mathcal{S}_{A \cup B}$ and the remaining relation for the case of the complete topological group becomes $\mathcal{S}_{A \cup B} \subset \mathcal{S}_A + \mathcal{S}_B$. The next statement is a fragmented generalization for linear maps between vector spaces [AII.16 §1.7 Cor. 1]:

19 **Statement.** ***((Non-) Commutation of Morphism and Linear Sum)*** *A morphism $f : E \to E'$ of Hausdorff topological vector spaces $(E, +, \mathbb{K}, \mathcal{O})$ and $(E', +, \mathbb{K}, \mathcal{O}')$ with $A \subset E$, yields the following: If the topological group $(E', +, \mathcal{O}')$ is complete then*

$$f\left(\mathrm{sum}_{\mathbb{K},\mathcal{O}}(A)\right) \subset \mathrm{sum}_{\mathbb{K},\mathcal{O}'}\left(f(A)\right)$$

If the restriction $f|_A : A \to E'$ is injective, then the subset relation becomes an equality relation.

Proof Indication: An element $f\left(\sum_{a \in A} x_a \cdot a\right)$ inside the left side of the subset relation is equal to $\sum_{a \in A} x_a \cdot f(a)$ by [TGIII.41 §5.5 Prop. 5] and the presupposed continuity. If the map f is not injective on A, then partition the family $(x_a)_{a \in A}$ to sets of scalars with indices that are mapped to the same value: $\{\{x_a | f(a) = b\} | b \in f(A)\}$. Summability of subfamilies [TGIII.39 §5.3 Prop. 2] and reassociation [TGIII.41 §5.3 Th. 2], needing the completeness of the topological group $(E, +, \mathcal{O})$, giving $\sum_{b \in f(A)} \left(\sum_{f(a)=b} x_a\right) \cdot b$ yields an element in $\mathrm{sum}_{\mathbb{K},\mathcal{O}'}(f(A))$. But if the map f is injective on A, then follows directly $\sum_{a \in A} x_a \cdot f(a) = \sum_{b \in f(A)} x_b \cdot b$. ■

20 **Statement.** ***(Relation between Linear Sum and Span)*** *For a subset A of a Hausdorff topological vector space $(E, +, \mathbb{K}, \mathcal{O})$ follows:*

$$\mathrm{span}(A) \subset \mathrm{sum}_{\mathcal{O}}(A) \subset \mathrm{cl}_{\mathcal{O}}(\mathrm{span}(A))$$
$$A \text{ is finite } \Rightarrow \mathrm{span}(A) = \mathrm{sum}_{\mathcal{O}}(A)$$

Proof Indication: Linear combinations are linear sums with finitely many vectors. To see the second subset relation, consider an element $x \in \mathrm{sum}(A)$, meaning that the element $x = \lim_{F \in \mathcal{F}} \sum_{a \in F} x_a \cdot a$ is a limit of a net of linear combinations given by the set $\mathcal{F} := \{F \,|F \subset A \,\mathrm{and}\, F \,\mathrm{is\,finite}\}$ which is directed[8] by the inclusion relation. Since all elements of the net are inside the linear span $\mathrm{span}\,(A)$ of the set A and each neighborhood of the element x contains some of them, the definition of the closure gives $x \in \mathrm{cl}_{\mathcal{O}}\,(\mathrm{span}\,(A))$. ■

Taking the $\mathcal{O}$-closure, here indicated by an over-line, of the subset relations above gives the following:

$$\mathrm{sum}\,(A) = \overline{\mathrm{sum}\,(A)} \Rightarrow \overline{A} \subset \mathrm{sum}\,(A) = \overline{\mathrm{span}\,(A)}$$

Generally, a linear sum and the closure of the linear span are not equal as can be seen in the following example:

21 **EXAMPLE.** ***(Span Closure and Linear Sum)*** *Consider the field of the real numbers, together with the topology which is given by the standard absolute value:* $\left(\mathbb{R}, \mathcal{O}_{|.|}\right)$. *Consider a compact subset, like the interval* $[-1,1] \subset \mathbb{R}$. *Now consider the set* $B\,([-1,1],\mathbb{R})$ *of all bounded functions* $f : [-1,1] \to \mathbb{R}$, $\exists b(b > 0 \,\mathrm{and}\, \forall x(x \in [-1,1] \Rightarrow \mid f(x) \mid < b))$ *on this interval into the set of all real numbers. Then, that space is a vector space with a norm* $\|f\| := \sup_{\mathrm{x}\in[-1,1]}\,(|f(x)|)$, *even a normed algebra, with respect to operations defined pointwise. Thus the algebra is a norm-topological algebra,*

$$\left(B\,([-1,1],\mathbb{R})\,, +, \cdot, \mathcal{O}_{||.||}\right)$$

which is automatically first countable and Hausdorff. [TGX §3.2 fourth and fifth sentences, TGX §3.2 facing Prop. 4 and TGIX §3.7 Expl. 3] (Since the additive topological group of the real numbers is complete, so is the norm topological algebra. [TGX §3.2 Prop. 3]) In this algebra define the subset

$$A := \{m_n \,|m_n : \,[-1,1] \to \mathbb{R}, x \mapsto x^n \,\mathrm{and}\, n \in \mathbb{N}_0\} \subset B\,([-1,1],\mathbb{R})$$

of all monomial functions. The definition [TGX §4.1] of uniform approximation by elements of A, *that is given by the norm-topology, and Stone's Theorem [TGX §4.2 Th. 2 (Stone)], for the space of all* $\mathcal{O}_{[-1,1]}-\mathcal{O}_{\mathbb{R}}$*-continuous functions* $C([0,1],\mathbb{R}) \subset B\,([-1,1],\mathbb{R})$, *yield:*

$$\mathrm{cl}_{\mathcal{O}_{||.||}}\,(\mathrm{span}_{\mathbb{R}}\,(A)) = C([-1,1],\mathbb{R})$$

Contrastingly, the linear sum $\mathrm{sum}_{\mathcal{O}_{||.||}}\,(A)$ *of the monomial functions contains functions* f, *which can be written as a* $\mathcal{O}_{||.||}$*-sum of infinitely many monomial elements,* $f = \sum_{a\in A} x_a \cdot a = \sum_{n\in\mathbb{N}_0} x_n \cdot m_n$ *over a family of scalars* $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}_0}$. *Therefore the map has to be real analytic, within the interval [-1,1], with* $n!\,x_n = f^{(n)}(0)$, *were* $f^{(n)}$ *is the* n*-th derivative of the function. Since there are continuous functions which are not real analytic within the interval [-1,1], for example the map* $|\;.\;| : [-1,1] \to \mathbb{R}, x \mapsto y := \begin{cases} x & \mathrm{if}\, x \geq 0, \\ -x & \mathrm{if}\, x < 0. \end{cases}$ *of the restricted absolute value, the following has been shown:*

$$\mathrm{cl}_{\mathcal{O}_{||.||}}\,(\mathrm{span}_{\mathbb{R}}\,(A)) \neq \mathrm{sum}_{\mathcal{O}_{||.||}}\,(A)$$

[8] A *directed set* is a set X with a relation $\leq$; that satisfies reflexiveness: $\forall x\,(x \in X \Rightarrow x \leq x)$, transitivity $\forall x,y,z\,(x,y,z \in X \,\mathrm{and}\, x \leq y \,\mathrm{and}\, y \leq z \Rightarrow x \leq z)$ and each pair of element is related to a third one: $\forall x,y\,(x,y \in X \Rightarrow \exists z\,(z \in X \,\mathrm{and}\, x \leq z \,\mathrm{and}\, y \leq z))$

Summability in Vector Quotient Spaces

Spaces with algebraic structures like that of a group or a vector space, can be partitioned by cosets of respective ideals; on these partitions often can be defined the same algebraic structures like that of the original space. For vector spaces see [AII.7 §1.3 sentence before Example 6)].

Additive groups and especially vector spaces may be identified as being isomorphic to a direct sum space. Such an additional internal structure often makes the elements of such a space more easily describable; see [AII.17 §1.8 Def. 6] [AII.19 §1.9 Def. 8]. Generally, these spaces allow easier accessible characterizations of quotient space [AII.20 §1.9 Prop. 13].

The topologically extended counterparts for quotient groups [TGIII.13 §2.6 Prop. 16] (for vector spaces [EVTI.5 §1.3]) and direct sum groups [TGIII.46 §6.2 Def. 1] (for direct sum vector spaces [TVSI.19 §2.1 last sentence]) involve heavily the quotient topology [TGI.20 §3.4 Def. 3] and continuity arguments.

For example ([TVSI.19 §2.1 last sentence] or similarly [EVTII.75 §4.5 Def. 2]) a topological vector space $(E,+,\mathbb{K}\,\mathcal{O})$ is called a *topological direct sum* $E = F \oplus_{\text{top}} G$ *of two subspaces* $F, G \leq_{\text{vector}-\text{space}} E$ if and only if the direct sum[9] space

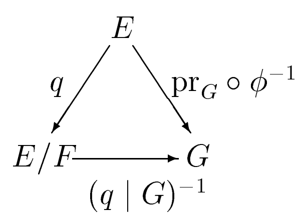
$$F \bigoplus G = \{(f,g)\,|\, f \in F \,\text{and}\, g \in G\} = F \times G$$

of the two subspaces is isomorphic to the original vector space; that means there exists a bijective, linear, continuous and open map $\phi : F \bigoplus G \to E$, $(f,g) \mapsto f+g$. There the homeomorphy is meant with respect to the product topology composed of the topologies of the subspaces.

A generalization of [AII.20 §1.9 Prop. 13] is the following observation:

22 **Statement.** *Consider a topological vector space* $(E,+,\mathbb{K}\,\mathcal{O})$ *which is also a topological direct sum* $E = F \oplus_{\text{top}} G$ *of two subspaces* $F, G \leq_{\text{vector}-\text{space}} E$*. And let the topological vector quotient space* $\left(E/F,+,\mathbb{K},\mathcal{O}_{E/F}\right)$ *furnish the quotient morphism* $q : E \to E/F$, $x \mapsto x+F$*. Then the restriction* $q \mid G : G \to E/F$ *of the quotient morphism is an isomorphism of topological vector spaces (linear, continuous and open)*

Proof Indication: By [AII.20 §1.9 Prop. 13] the restriction $q \mid G$ is an isomorphism of vector spaces. The definition of the quotient topology makes the quotient map continuous and thereby also its restriction to the subspace G. By $E = F \oplus_{\text{alg}} G$, there are elements $x = f + g$ so that follows $(q \mid G)^{-1} \circ q(x) = g$ and with $\text{pr}_G \circ \phi^{-1}(x) = g$ giving the commutative diagram:



[9] In the case of only two or finitely many subspaces the direct sum space $\bigoplus_{j\in J} F_j$ is equal to the product space $\prod_{j\in J} F_j$, the two kinds of spaces differ only when the index set contains infinitely many elements.

The product topology on $F \oplus G$ makes the projection $\mathrm{pr}_G : F \times G \to G$ continuous and the isomorphism ϕ is considered as open, so the map $\mathrm{pr}_G \circ \phi^{-1}$ is continuous. This, together with the final character of the quotient topology [TGI.14 §2.4 Prop.6 and Expl. I], then makes the inverse $(q \mid G)^{-1}$ continuous. This, finally, makes the map $q \mid G$ an isomorphism of topological vector spaces. ■

The goal is to relate the summability in an original space to the summability in the quotient space:

23 **STATEMENT.** ***(Summability in Quotient Space)*** *Consider a direct topological sum* $E = F \oplus_{\text{top}} G$ *and a family* $(g_j)_{j\in J} \in G^J$ *of vectors inside one of the subspaces, then follows:*

$$\sum_{j\in J} g_j = 0 \Leftrightarrow \sum_{j\in J} (g_j + F) = F \in E/F$$

Proof Indication: "⇒" If $\sum_{j\in J} g_j = 0$ apply the quotient map and use its continuity to see the result. "⇐" For a family $(g_j)_{j\in J} \in G^J$of vectors in the subspace G, $\sum_{j\in J} g_j = 0$ in E is equivalent to the same relation in G. Since the map $q \mid G$ is an isomorphism of topological vector spaces, and since the family $(g_j)_{j\in J}$ contains only elements in G, there is a one-to-one relation between g_j and $g_j + F$ $(j \in J)$, 0 and F. Thus applying its continuous inverse $(g \mid G)^{-1}$ to the sum $\sum_{j\in J} (g_j + F) = F \in E/F$ in the quotient space, yields a sum $\sum_{j\in J} g_j = 0$ in G, a subspace of the original space E. Since injections are continuous, the sum relation remains valid in the original space E. ■

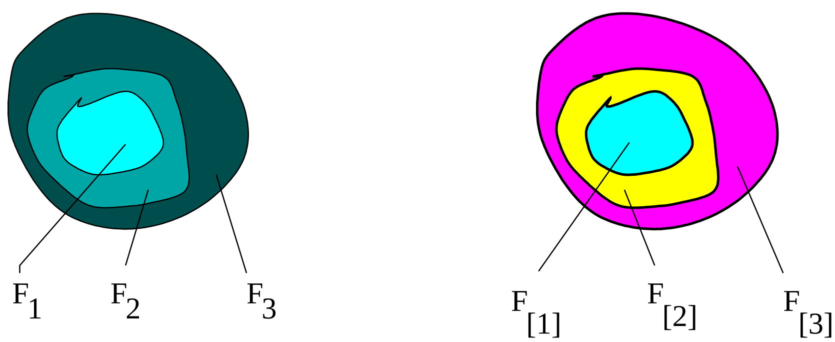
Algebraic-, Topological- and Reduced Basis

Direct sums or product spaces can be used to describe the internal structure of a vector space. Direct sums are called graded, if the index set is a commutative monoid [AII.163 §11.1][ACIII.156 §1.2]. Often vector spaces are defined by such structures in the first place.

Product spaces of copies of the given field make it possible to describe vectors by families of scalars. Different ways of doing this give the different notions of bases. A subset of a vector space is called algebraic basis exactly if it is the index set of a direct sum space of fields, so that the direct sum is isomorphic to the original vector space [AII.25 §1.11 Def. 10]. Two other notions are the known topological basis and the so-called *reduced basis* in Hausdorff topological vector spaces.

Algebraic Free Sets and Gradations

Generally, a filtration inside a set X [LIEII.38 §4.1 Def. 1] is a totally ordered[10] sequence $(F_n)_{n\in N}$ of subsets $F_n \subset X$, where the index order matches the containment order $n < m \Rightarrow F_n \subset F_m$ of the subsets. Usually the total order is given by an index set N which is also a commutative monoid, like the natural numbers $(\mathbb{N}, +)$ or the real numbers.



Consider an element $x \in X$ and the minimum[11] $\min_N(\{n \,|n \in N \,\text{and}\, x \in F_n\})$; if the ar-

[10] Elementary properties of relations on a set X are: (R) reflexive $\forall x\,(x \in X \Rightarrow x \leq x)$ (T) transitive $\forall x, y, z\,(x, y, z \in X \,\text{and}\, x \leq y \,\text{and}\, y \leq z \Rightarrow x \leq z)$ (A) antisymmetry $\forall x, y\,(x, y \in X \,\text{and}\, x \leq y \,\text{and y}\, \leq x \Rightarrow x = y)$ (S) symmetry $\forall x, y\,(x, y \in X \Rightarrow (x \leq y \Leftrightarrow y \leq x))$ (TT) totality $\forall x, y\,(x, y \in X \Rightarrow x \leq y \,\text{or}\, y \leq x)$ Sets with a relation $\leq$ that satisfies (R, T) are called pre-ordered, (R, A, T) order, (R, S, T) equivalence relation, (R, A, T, TT) total order.

[11] Consider a set X with a pre-order relation (R, T). An element $s \in X$ is called a lower bound of a subset $U \subset X :\Leftrightarrow \forall u\,(u \in U \Rightarrow s \leq u)$. Denote here $\mathrm{L}_X(U)$ the set of all lower bounds of a subset $U \subset X$. Observe that $\mathrm{L}_X(\emptyset) = X$. An element $s \in X$ is called a minimum (written $\min_X(U)$) of a subset $U \subset X :\Leftrightarrow$

gument is empty, the minimum is not defined. This case, where elements $x \in X \backslash \bigcup_{n \in N} F_n$ are not inside the filtration, can be included in several ways[12]. The least invasive way seems to use the naturally ordered set $N \cup \{\infty\}$, with $\forall n\,(n \in N \Rightarrow n \leq \infty)$, introducing an element ∞ as the maximum of the set $N \cup \{\infty\}$. Then the following term always is defined:

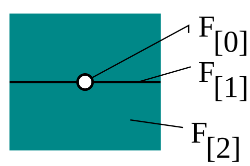
$$\mathrm{v}(x) := \inf_{N \cup \{\infty\}} (\{n\,|n \in N \text{ and } x \in F_n\}) \qquad x \in X$$

24 **DEFINITION** ***(Order Function and Level Sets)*** *For a set X, with a totally ordered set $(N, \leq)$ indexing a filtration $(F_n)_{n \in N}$, consider the map $\mathrm{v} : X \to N \cup \{\infty\},\ x \mapsto \mathrm{v}(x)$ into the totally ordered set $(N \cup \{\infty\}, \leq)$ with a maximal element ∞. This map is called the* order function of the filtration $(F_n)_{n \in N}$. *The inverse image $\mathrm{v}^{-1}(n)$ of the element $n \in N$ is called n*-th level set of the filtration*: $G_{[n]} := \mathrm{v}^{-1}(n)$ ($n \in N \cup \{\infty\}$). Idea from [LIEII.38 §4.2]*

Note that unlike the filtration, the set $\left\{F_{[n]}\,|n \in N \cup \{\infty\}\right\}$ of inverse images of the order function partitions the original set X. And if the totally ordered set $(N \cup \{\infty\}, \leq)$ with a maximal element has a subset $N_\subset \subset N$ that permits an operation $\mathrm{pred} : N_\subset \to N,\ n \mapsto \mathrm{pred}(n)$ with $\forall n'\,(n' \in N \Rightarrow n' \leq \mathrm{pred}(n) \text{ or } n \leq n')$, then the level sets can be described as follows:

$$F_{[n]} = F_n \backslash F_{\mathrm{pred}(n)} = \mathrm{v}^{-1}(n) = \{x\,|x \in X \text{ and } \mathrm{v}(x) = n\} \qquad n \in N_\subset$$

25 **EXAMPLE.** *1) Consider the two-dimensional space $\mathbb{R} \times \mathbb{R} = \bigoplus_{i=1}^{2} E_i$, with $E_1 = E_2 = \mathbb{R}$ which gives a filtration: $F_2 := \mathbb{R} \times \mathbb{R}$, $F_1 := \mathbb{R} \times \{0\}$, and with $F_0 := \{0\} \times \{0\}$. Then there occur level sets: $F_{[0]} = \{0\} \times \{0\}$, $F_{[1]} = F_1 \backslash F_0 = (\mathbb{R} \backslash \{0\}) \times \{0\}$ and $F_{[2]} := F_2 \backslash F_1 = (\mathbb{R} \times \mathbb{R}) \backslash (\mathbb{R} \times \{0\})$.*



2) Consider a vector space $(E, +, \mathbb{K})$ with a family of subspaces $(E_n)_{n \in \mathbb{N}_0}$ so that the vector space $E \cong_{\mathrm{vector-space}} \bigoplus_{n \in \mathbb{N}_0} E_n$ is isomorphic to a direct sum space (with an isomorphism $\phi : \bigoplus_{n \in \mathbb{N}_0} E_n \to E, (x_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} x_n$), meaning the vector space is graded. Each graded vector space has a filtration $(F_n)_{n \in \mathbb{N}_0}$ isomorphic to $(\bigoplus_{k=0}^{n} E_k)_{n \in \mathbb{N}_0}$ ($F_0 = \{0\}$). The order map $\mathrm{v} : E \to \mathbb{N}_0,\ x \mapsto \mathrm{v}(x)$, $\mathrm{v}(x) = \min(\{n\,|n \in \mathbb{N}_0 \text{ and } x \in F_n\})$ together with the (idempotent) projection $\mathrm{p}_n : E \to E,\ \mathrm{p}_n := \phi\,|_{E_n} \circ \mathrm{pr}_n \circ \phi^{-1}$(idempotent $\mathrm{p}_n^2 = \mathrm{p}_n$ with an image $\mathrm{im}(\mathrm{p}_n) = E_n$) make it possible to rewrite the level sets as $F_{[k]} = \{x\,|x \in F_n \text{ and } \mathrm{p}_n(x) \neq 0\} \cong \bigoplus_{k=0}^{n} E_k \backslash \bigoplus_{k=0}^{n-1} E_k$ ($k \in \mathbb{N}$) and $F_{[0]} = \{0\}$.

$\forall u\,(u \in U \Rightarrow s \leq u)$ and $s \in U$. An order relation (RAT) makes a minimum unique and $\min_X(\emptyset)$ does not exist. The maximum of the set of all lower bounds is called infimum: $\inf_X(U) := \max_X(\mathrm{L}_X(U))$. Here $\inf_X(\emptyset) = \max_X(X)$.

[12] Consider $x \in X$. (1) Just consider the cases explicitly: $\mathrm{v}(x) := \begin{cases} \min_N(\{n\,|n \in N \text{ and } x \in F_n\}) & \text{if } x \in \bigcup_{n \in N} F_n \\ \infty & \text{if } x \in X \backslash \bigcup_{n \in N} F_n \end{cases}$ (2) Define $F_\infty := X$ and a maximal element ∞, with $\forall n\,(n \in N \Rightarrow n \leq \infty)$, then consider the minimum $\min_{N \cup \{\infty\}}(\{n\,|n \in N \cup \{\infty\} \text{ and } x \in F_n\})$ (3) Consider the infimum as shown above; in that second case, the relation $\inf_X(\emptyset) = \max(X)$ can be applied without leaving anything undefined. Here the filtration needs no modification.

A kind of set, with sets only faintly resembling level sets, permits an interesting conclusion:

26 **Statement.** ***(Unique Summands)*** *Consider the Example 2) together with a family of subspaces $A_n \leq F_n$ and the intersection property $A_n \cap F_{n-1} = \{0\}$ ($n \in \mathbb{N}$).*

(i) The only element $a = (a_n)_{n\in\mathbb{N}}$ of the direct sum space $\bigoplus_{n\in\mathbb{N}} A_n$ that can be summed to zero is the zero: $\sum_{n\in\mathbb{N}} a_n = 0 \Rightarrow \forall n\,(n \in \mathbb{N} \Rightarrow a_n = 0)$

(ii) There exists an injective linear map: $f : \bigoplus_{n\in\mathbb{N}} A_n \to E,\ (a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n$.

Proof Indication: (i) Consider an element $a \in \bigoplus_{n\in\mathbb{N}} A_n$ with a finite support, therefore define $N := \max(\mathrm{supp}\,(a))$ and index the elements of the support according to their order: $s_1 < \ldots < s_{N-1} < s_N$. Presupposedly the sum $0 = \sum_{n\in\mathbb{N}} a_n = \sum_{n=1}^{N} a_{s_n}$ is zero, which can be rewritten as $a_{s_N} = -\sum_{n=1}^{N-1} a_{s_n}$. From the choice of a follows $a_{s_N} \in A_{s_N}$ and since $A_n \leq F_n$ both are subspaces, follows $-\sum_{n=1}^{N-1} a_{s_n} \in F_{s_{N-1}} \subset F_{s_N-1}$. So, by the intersection property, follows $a_{s_N} = 0$. Iterating that argument leaves $a = 0$. (ii) Definedness and linearity of the map should be elementary. The injectivity follows by Statement (i). ■

In the next statement is used a notion of algebraic freedom of sets:

27 **Definition** ***(Algebraic Freedom of Sets of Sets)*** *In a given vector space E, a* set $\mathcal{S}$ of subsets $S \subset E$ is called algebraically free *if and only if*

$$\mathrm{span}\,(\cup\mathcal{S}) \cong_{\mathrm{vector-space}} \bigoplus_{S\in\mathcal{S}} \mathrm{span}\,(S)\ .$$

Whereby, individual sets in $\mathcal{S}$ need not to be algebraically free. The case in which the elements of the set $\mathcal{S}$ are algebraically free yields:

28 **Statement.** ***(Compounded Algebraic Freedom)*** *In a vector space E consider a set $\mathcal{S}$ of subsets $S \subset E$ which is algebraically free and in which all elements $S \in \mathcal{S}$ are algebraically free. Then the union $\bigcup\mathcal{S}$ is algebraically free.*

Proof Indication: The definition of algebraic freedom of sets of sets, above gives $\mathrm{span}\,(\cup\mathcal{S}) \cong_{\mathrm{vector-space}} \bigoplus_{S\in\mathcal{S}} \mathrm{span}\,(S)$. Since every subset $S \in \mathcal{S}$ is algebraically free, each set S is a basis of the span $\mathrm{span}(S)$. Together with the definition of an algebraic basis and the isomorphy of direct sum spaces $\bigoplus_{S\in\mathcal{S}} \mathrm{span}\,(S) \cong \bigoplus_{s\in S\in\mathcal{S}} \mathbb{K}$ follows that the union $\bigcup\mathcal{S}$ is algebraically free in the original vector space. ■

The preparations before make it possible to formulate a variant and generalization of [AII.26 §1.1 Prop. 19]:

29 **Statement.** ***(Algebraic Freedom with Level Sets)*** *Consider a vector space $(E, +, \mathbb{K})$ over a field $\mathbb{K}$ and let the vector space be a direct sum space $E \cong \bigoplus_{n\in\mathbb{N}_0} E_k$ of vector subspaces $E_k \leq E$. And consider the (idempotent) projection $\mathrm{p}_n : E \to E$, $\mathrm{p}_n^2 = \mathrm{p_n}$ onto the subspace E_n. (See also Example 2) on page 30.) — Now consider a subset $B \subset E$ and let it intersect the level sets $B_{[n]} := B \cap F_{[n]}$, ($n \in \mathbb{N}_0$); consequences are:*

(Partition) The family $\left(B_{[n]}\right)_{n\in\mathbb{N}_0}$ partitions the set B.

(Contain) $\forall n \left(n \in \mathbb{N} \Rightarrow \mathrm{p}_n\left(B_{[n]}\right) \text{ is algebraically free}\right) \Rightarrow \mathrm{span}\left(B_{[n]}\right) \subset F_{[n]} \cup \{0\}$

(Free Set) $\forall n \left(n \in \mathbb{N} \Rightarrow \left(\mathrm{span}\left(B_{[n]}\right) \subset F_{[n]} \cup \{0\} \Rightarrow \left\{ B_{[n]} \middle| n \in \mathbb{N}\right\} \text{ is algebraically free}\right)\right)$

(Free) $\forall n \left(n \in \mathbb{N} \Rightarrow \mathrm{p}_n\left(B_{[n]}\right) \text{ is algebraically free}\right) \Rightarrow B \text{ is algebraically free}$

Proof Indication: (Partition) Since the set $\left\{ F_{[n]} \middle| n \in \mathbb{N}_0 \right\}$ is a partition of the original space E, intersections $\left\{ B \cap F_{[n]} \middle| n \in \mathbb{N}_0 \right\}$ with subsets $B \subset E$ partition the subsets. (Contain) For all indices $n \in \mathbb{N}$ of images $\mathrm{p}_n\left(B_{[n]}\right)$, being algebraically free, follows that the kernel $\ker\left(\mathrm{p}_n\big|_{\mathrm{span}\left(B_{[n]}\right)}\right) = \{0\}$ is the singleton. So, for all elements $x \in \mathrm{span}\left(B_{[n]}\right) \setminus \{0\}$, the injectivity of the restricted (idempotent) projection yields the non-equality $\mathrm{p}_n(x) \neq 0$; meaning $x \in F_{[n]}$; or $\mathrm{span}\left(B_{[n]}\right) \subset F_{[n]} \cup \{0\}$. (Free Set) For all indices $n \in \mathbb{N}$, define subspaces $A_n := \mathrm{span}\left(B_{[n]}\right) \leq E$. From the inclusions $B_{[n]} \subset F_{[n]} \subset F_n \leq E$ follows the relation $A_n \leq F_n$. And since presupposedly the linear span is a subset $\mathrm{span}\left(B_{[n]}\right) \subset F_{[n]} \cup \{0\}$ of a level set plus the zero, follows that the intersection $A_n \cap F_{n-1} = \{0\}$ gives the singleton. Applying the Statement 26 about the unique summands and using that morphic injections are isomorphic on their images yields the isomorphy $\bigoplus_{n\in\mathbb{N}} \mathrm{span}\left(B_{[n]}\right) \cong \mathrm{span}\left(\cup\left\{ B_{[n]} \middle| n \in \mathbb{N}\right\}\right)$. By the Definition 27 of the freedom of sets of sets follows that the set $\left\{ B_{[n]} \middle| n \in \mathbb{N}\right\}$ is algebraically free.
(Free) First, note that as presupposed, if the image $\mathrm{p}_n\left(B_{[n]}\right)$ is algebraically free, then the argument $B_{[n]}$ is also algebraically free. Secondly, use (ii) and then apply (iii) to get $\left\{ B_{[n]} \middle| n \in \mathbb{N}\right\}$ an algebraically free set of sets. Finally, the Statement 28 about compounded algebraic freedom makes the union $\bigcup_{n\in\mathbb{N}} B_{[n]} = B$ algebraically free. ■

Reduced Basis and Topological Basis

Since limits in Hausdorff topologies always are found uniquely [TGI.52 §8.1 Prop. 1], all topological spaces in the following are taken to be Hausdorff.

30 **Definition** ***(Summing a Space, Reduced Freedom, Reduced Basis)*** *In a Hausdorff topological vector space* $(E, +, \mathbb{K}, \mathcal{O})$ *over a Hausdorff topological field*[13] $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$ *a subset* A *can have the following named properties:*

(LS) The subset A linearly sums, or $\mathcal{O}$-sums, the vector space *if and only if the linear sum of the subset contains the vector space:* $E \subset \mathrm{sum}_{\mathcal{O}}(A)$

(RF) The subset A *is called* ($\mathcal{O}$-) reduced free *if and only if any linear sum of a family of scalars that* A*-sums the zero must be zero:*
$\forall \underline{s} \left(\underline{s} \in \mathbb{K}^A \text{ and } (s_a \cdot a)_{a\in A} \text{ is } \mathcal{O}-\text{summable} \Rightarrow \left(\sum_{a\in A} s_a \cdot a = 0 \Rightarrow \underline{s} = \underline{0}\right)\right)$ *where* $\underline{s} := (s_a)_{a\in A}$

[13] That is a weak restriction since every nontrivially topologized field is Hausdorff [TGIII.55 last sentence of §6.7]. And valued fields, which usually occur in applications, are Hausdorff anyway.

(RB) The subset A is called a reduced basis of the topological vector space *if and only if the subset linearly sums the vector space (LS) and if the subset is reduced free (RF).*

Here nets of partial sums (successively enlarged linear combinations) approximate a limit element. In analogy to the algebraic basis B of a vector space E, which can be characterized by an unique isomorphism $E \cong \mathbb{K}^{(B)}$, the notions of a reduced basis also can be described by a linear map:

31 **NOTE.** ***(Reduced Basis and Linear Map)*** *Retaining the presuppositions of the definition above, consider a map $\varphi_A : \$ \to E$ as introduced in Statement 17 on page 24: Then note the following equivalences: (LS) $\Leftrightarrow$ φ_A is surjective; (RF) $\Leftrightarrow$ φ_A is injective and finally (RB) $\Leftrightarrow$ φ_A is bijective. And observe that by Statement 20 on page 25, for a set A that linearly sums E (or for a reduced basis A), the linear span is dense $\mathrm{cl}_{\mathcal{O}}(\mathrm{span}(A)) = E$ in the original vector space.*

Observe that the linear map φ_A needs NO topological properties! The definition of a reduced basis differs from that of an topological basis which is defined in the same environment:

32 **DEFINITION** ***(Totality, Topological Freedom, Topological Basis)*** *In a Hausdorff topological vector space $(E, +, \mathbb{K}, \mathcal{O})$ over a Hausdorff topological field $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$ a subset A can have the following named properties:*

(T) The subset A is called total subset of the topological vector space *if and only if the topological closure of its linear span contains the vector space: $E \subset \mathrm{cl}_{\mathcal{O}}(\mathrm{span}(A))$*

(TF) The subset A is called ($\mathcal{O}$-) topologically free subset of the vector space *if and only if any element of the subset is not inside the closed linear span of the rest of the elements: $\forall a\,(a \in A \Rightarrow a \notin \mathrm{cl}_{\mathcal{O}}(\mathrm{span}(A \backslash \{a\})))$*

(TB) The subset A is called a topological basis of the topological vector space *if and only if the subset is total in the vector space (T) and if the subset is topologically free (F)*

[EVTI.15 §2.1 Def. 1 for (T); EVTI.16 §2.1 Def. 2 for (F)]

Contrary to the case of the reduced basis, nets of linear combinations which approximate limit elements are not restricted to being only partial sums. A corresponding characterization of these notions by means of a linear map $\sigma_A : \mathbb{K}^{(A)} \to E$, $\sigma_A\left((x_a)_{a\in A}\right) := \sum_{a\in A} x_a \cdot a$ yields that the set A satisfies the equivalences: (T) $\Leftrightarrow$ σ_A has a dense image; (TF) $\Leftrightarrow$ for all elements $a \in A$, the restriction $\sigma_A\,|_{\mathbb{K}^{(A\backslash\{a\})}}$ has no dense image (where the set $\mathbb{K}^{(A\backslash\{a\})}$ represents the isomorphic subset in $\mathbb{K}^{(A)}$). As can be seen in the first of the following examples, the two basis notions are not equivalent:

33 **EXAMPLE.** (Δ) *Reconsider the Example 21, on page 26, about the comparison of a topologically closed linear span and a linear sum. There, $B([-1,1], \mathbb{R})$, the topological vector space of all bounded real-valued functions on a compact interval of the real numbers, has been considered the set $A = \{x \mapsto x^n \mid n \in \mathbb{N}_0\}$ of all monomial functions. This set A is an algebraic basis of the vector subspace of all polynomial functions. By the uniqueness of the Taylor-expansion with respect to a chosen base point (here the zero), the set A is*

a reduced basis of the vector subspace of all functions that permit such an expansion[14]*. And finally, as could also be seen in the referenced example the set A is total in the vector subspace $C([-1,1],\mathbb{R})$ of all continuous functions. But the set of all monomial functions is no topological basis of this vector subspace, because in [TGX §4.2 Lemma 2] is shown that there are monomials which can be approximated by polynomials.*

(product) For a set B and a Hausdorff topological field $(\mathbb{K},\mathcal{O}_{\mathbb{K}})$, consider the Hausdorff topological vector space $\left(\mathbb{K}^B,+,\mathbb{K},\mathcal{O}_\pi\right)$, which is given by operations defined componentwise and the product topology $\mathcal{O}_\pi$ on the product space $\mathbb{K}^B$ (Statement 9 on page 18). By Statement 10 (on page 18) about summability in product topologies, the subset of singly supported elements, given by the set B, is a reduced basis.

(box) For a set B and a Hausdorff topological field $(\mathbb{K},\mathcal{O}_{\mathbb{K}})$, consider the Hausdorff topological vector space $\left(\mathbb{K}^{(B)},+,\mathbb{K},\mathcal{O}\right)$, which is given by operations defined componentwise; the topology $\mathcal{O}:=\mathcal{O}_{\text{box}}\cap\mathbb{K}^{(B)}$ is derived from the box topology $\mathcal{O}_{\text{box}}$ on the product space $\mathbb{K}^B$. By the Statements 11 (on page 18) and 12 about box topologies, the subset of singly supported elements, given by the set B, is an algebraic basis and simultaneously a reduced basis.

(Hilbert) The maximal orthonormal system of a Hilbert space is a reduced basis and also a topological basis. [EVT V.149 §2.2 Cor. 1]

((Schauder)) A Schauder basis B of a separable Banach space E may not be a reduced basis, because sequences of partial sums are used for the approximation of elements in E by elements from $\operatorname{span}(B)$, while the summability of families $(r_b\cdot b)_{b\in B}$, $r_b\in\mathbb{R}$ is a much stronger condition. [TVSIII.114 past Cor. of 9.5 (TVSII.41 facing 2.1 for the definition of (B)-space)]

Some Properties of the various Basis-related Notions

34 **STATEMENT.** ***(Unfreedom)*** *A subset A in a Hausdorff topological vector space $(E,+,\mathbb{K},\mathcal{O})$ over a Hausdorff topological field $(\mathbb{K},\mathcal{O}_{\mathbb{K}})$ is* not reduced free *if and only if one of its elements is a linear sum of the remaining elements:*

$$A \text{ not reduced free } \Leftrightarrow$$
$$\exists a_0,\,(s_a)_{a\in A\setminus\{a_0\}}\left(a_0\in A\text{ and }(s_a)_{a\in A\setminus\{a_0\}}\in\mathbb{K}^{A\setminus\{a_0\}}\text{ and }\textstyle\sum_{a\in A\setminus\{a_0\}}s_a\cdot a=a_0\right)$$

The negation of the reduced freedom looks very much like the above, nevertheless there is still something to be shown:

Proof Indication: “⇒” The negation of the item (RF) of Definition 30 yields a nonzero family $(s_a)_{a\in A}\in\mathbb{K}^A$ of scalars which is A-summable so that $\sum_{a\in A}s_a\cdot a=0$. From the non-empty support of the family choose an element $a_0\in A$ and since summability remains valid if finitely many summands are missing and the continuity of the scalar multiplication, reformulate the sum expression to $a_0=\sum_{a\in A\setminus\{a_0\}}-\frac{s_a}{s_{a_0}}\cdot a$. “⇐” Now consider the right side of the stated equivalence above. The sum expression can be changed directly to $\sum_{a\in A\setminus\{a_0\}}s_a\cdot a-a_0=0$, which by [TGIII.41 §5.3 Prop. 3] can be

[14] The notion of a reduced basis therefore can be seen as a generalization of the notion of analyticity of functions to the level of a pure topological vector-space.

written as a sum $\sum_{a\in A} t_a \cdot a = 0$ of a non-trivial and A-summable family $(t_a)_{a\in A}$ of scalars given by $t_a := \begin{cases} 1 & \text{if } a = a_0, \\ s_a & \text{if a} \in A\backslash\{a_0\}. \end{cases}$ ■

35 **STATEMENT.** ***(On Freedom)*** *Again consider a Hausdorff topological vector space $(E, +, \mathbb{K}, \mathcal{O})$ over a Hausdorff topological field $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$:*

($\subset$) Taking the subset conserves any kind of freedom. [EVTI.16 §2.1 sentences before the last example]

(AF) A subset A is algebraically free if and only if $\forall a\,(a \in A \Rightarrow a \notin \operatorname{span}(A\backslash\{a\}))$

(RF) A subset A is reduced free if and only if $\forall a\,(a \in A \Rightarrow a \notin \operatorname{sum}_{\mathcal{O}}(A\backslash\{a\}))$

(TF) A subset A is topologically free if and only if $\forall a\,(a \in A \Rightarrow a \notin \operatorname{cl}_{\mathcal{O}}(\operatorname{span}(A\backslash\{a\})))$

($\emptyset, \{\bullet\}$) The empty set and non-zero singletons (due to the Hausdorff property) are algebraically free, reduced free and topologically free.

(RF⇒AF) Reduced free sets are always algebraically free.

(TF⇒RF, AF) Topologically free sets are always reduced free and algebraically free. [EVTI.16 §2.1 sentence before the last example]

Proof Indication: ($\subset$) The case of algebraic freedom follows from [AII.26 §1.11 Prop. 18]. To show the case for reduced freedom, consider a subset $A' \subset A$ and one of those A'-families $(s'_a)_{a\in A'} \in \mathbb{K}^{A'}$ of scalars that A'-sum the zero: $\sum_{a\in A'} s'_a \cdot a = 0$. Extend the given family of scalars to a new A-family $(s_a)_{a\in A} \in \mathbb{K}^A$ by defining $s_a := \begin{cases} s'_a & \text{if } a \in A', \\ 0 & \text{if a} \in A\backslash A'. \end{cases}$ By its definition, this family is A-summable to zero. Thus applying the original definition of reduced freedom to the family new A-family $(s_a)_{a\in A} \in \mathbb{K}^A$ follows that all scalars are zero especially those of the corresponding A'-family. In the case of the topological freedom, observe that the initial containment $A' \subset A$ is conserved by applying the linear span and the closure operators: $\operatorname{cl}_{\mathcal{O}}(\operatorname{span}(A'\backslash\{a\})) \subset \operatorname{cl}_{\mathcal{O}}(\operatorname{span}(A\backslash\{a\}))$ $(a \in A)$. Together with (TF) and the observation that non-inclusion is conserved by subsetting follows $\forall a\,(a \in A \Rightarrow \text{a} \notin \operatorname{cl}_{\mathcal{O}}(\operatorname{span}(A'\backslash\{a\})))$. Weakening the right side of this conclusion by restricting the elements to the subset A' gives the to be shown. (AF) Is stated in [AII.26 §1.11 Remark 1)] (RF) This is a reformulation of reduced unfreedom of Statement 34. (TF) This is the Item (TF) in the Definition 32 of the topological basis. ($\emptyset, \{\bullet\}$) Since false suppositions allow any conclusion, the three versions of freedom above ((AF), (RF) and (TF)) remain true for empty sets. For singletons the linear spans and linear sums in (AF), (RF) and (TF) above become the singleton $\{0\}$. This makes all non-zero singletons algebraically free and reduced free. Since the topological vector space is Hausdorff and therefore $\operatorname{cl}_{\mathcal{O}}(\{0\}) = \{0\}$, all non-zero singletons are even topologically free. (RF⇒AF) Writing down the original definition of reduced freedom (Definition 30, (RF)) and restricting it to families of scalars with finite support gives the algebraic freedom [AII.25 §1.11 Second sentence past Cor. 3]. (TF⇒RF, AF) Consider the original definition of topological freedom (Definition 32, (TF)) of a set A: $\forall a\,(a \in A \Rightarrow a \notin \operatorname{cl}_{\mathcal{O}}(\operatorname{span}(A\backslash\{a\})))$. Since $\operatorname{sum}_{\mathcal{O}}(A) \subset \operatorname{cl}_{\mathcal{O}}(\operatorname{span}(A))$ (Statement 20 on page 25.) follows the reduced freedom in the notation (RF) as shown above. Then the algebraic freedom is immediate. ■

A consequence of Zorn's lemma about the extensibility of algebraically free sets to algebraic bases [AII.95 §7.1 Th. 2] and the statement that every set being reduced free is also algebraically free, (RF⇒AF) yields: Every reduced basis can be added vectors until the resulting set is an algebraic basis. This motivated the name "reduced basis".

36 **Statement.** ***(Equivalences for a Reduced Basis)*** *Reconsider a Hausdorff topological vector space $(E, +, \mathbb{K}, \mathcal{O})$ over a Hausdorff topological field $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$ and the set $\mathcal{R} = \{A \mid A \subset E \,\mathrm{and}\, A \,\mathrm{is\, reduced\, free}\}$ of all reduced free sets in the vector space, together with the set $\mathcal{S} = \{A \mid A \subset E \,\mathrm{and}\, A \,\mathrm{sums}\, E\}$ of all subsets which linearly sum the vector space. As any set of subsets, these sets are partially ordered with respect to the inclusion relation. On this basis, the following statements are equivalent:*

(RB) The set B is reduced free and sums the original vector space $E \subset \mathrm{sum}\,(B)$.

(RB represent) For each element $x \in E$ in the original vector space E uniquely exists a B-summable family $(x_b)_{b\in B} \in \mathbb{K}^B$ of scalars and the linear sum $\sum_{b\in B} x_b \cdot b = x$ is equal to the given element.

(RB maximal) The set B is maximal in the set $\mathcal{R}$ of all reduced free subsets.

(RB minimal) The set B is minimal in the set $\mathcal{S}$ of all sets which sum the vector space $E \subset \mathrm{sum}\,(B)$.

Proof Indication: "(RB) ⇔ (RB represent)" This is immediate by Note 31 (on page 33) and its isomorphic case. "(RB) ⇒ (RB minimal)" Assume that the set B is *not* minimal in the set of all sets which sum E (show that B is *not* a reduced basis): There exists a set $C \subset B$ that sums the vector space $E \subset \mathrm{sum}\,(C)$ and each element $b \in B \backslash C$ can be written as a linear sum of elements in $C \subset B$. This means the set B is not reduced free. "(RB minimal) ⇒ (RB maximal)" Consider a reduced free set $D \in \mathcal{R}$ that is potentially larger $B \subset D$ than the set B. As presupposed, the set $B \in \mathcal{S}$ linearly sums $\mathrm{sum}\,(B) \supset E$ the vector space. Therefore the potentially larger reduced free set D is inside a linear sum: $\mathrm{sum}\,(B) \supset E \supset D$. Thus all elements in the set D are linear sums of elements in B, and with $B \subset D$ follows $D = B$. "(RB maximal) ⇒ (RB represent)" Consider a non-zero element $x \in E\backslash\{0\}$. Presupposedly the set B is maximal in the set of reduced free sets, therefore (unless $x \in B$) the union $B \cup \{x\}$ is not reduced free. This means that there exists a non-zero $B \cup \{x\}$-summable family $(s_b)_{b\in B\cup\{x\}} \in \mathbb{K}^{B\cup\{x\}}$ of scalars which sums the zero: $\sum_{b\in B\cup\{x\}} s_b \cdot b = 0$. Assume the scalar s_x to be zero, then due to the reduced freedom of the set B, all other scalars had to be zero, but this contradicts the original non-zeroness of the family. Using $s_x \neq 0$, [TGIII.41 §5.3 Prop. 3] about finite partitions of summable families and the continuity of the scalar multiplication changes the previous equation to: $x = \sum_{b\in B} -\frac{s_b}{s_x} \cdot b$. Finally, since the set B is reduced free, the linear sum is unique. ■

Reduced Basis and Direct Sums

An algebraic basis B of a vector space $(E, +, \mathbb{K})$, $E \cong \bigoplus_{b\in B} \mathbb{K}_b$ ($\mathbb{K}_b = \mathbb{K}$) automatically permits as much direct sum decompositions as there exist partitions of the basis [AII.12

§1.6 paragraph facing Cor. 1]. A reduced basis B of a topological vector space E, giving an isomorphism $\varphi_B : \mathbb{S}_B \to E$, $E \cong \mathbb{S}_B$ of a different kind (Definition 16 on page 23), has to be specially investigated:

37 **STATEMENT.** ***(Reduced Basis and Direct Sums)*** *Reconsider a Hausdorff topological vector space* $(E, +, \mathbb{K}, \mathcal{O})$ *over a Hausdorff topological field* $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$*. Now let the topological vector space be complete with respect to the uniformity*[15] *given by the uniformity of its topological group. And let the subset* $B \subset E$ *be a reduced basis. Then, for any subset* $C \subset B$ *of the reduced basis, there is a linear bijection* $\phi : \operatorname{sum}(C) \times \operatorname{sum}(B \backslash C) \to E$*,* $(x, y) \mapsto x + y$ *of vector spaces:*

$$\mathrm{E} \cong \operatorname{sum}(C) \oplus_{\mathrm{alg}} \operatorname{sum}(B \backslash C)$$

and the following diagram is commutative

$$\begin{array}{ccc} \mathbb{S}_B & \xrightarrow{\varphi_B} & E \\ \uparrow & & \uparrow \phi \\ \mathbb{S}_C \times \mathbb{S}_{B\backslash C} & \xrightarrow[\varphi_C \times \varphi_{B\backslash C}]{} & \operatorname{sum}(C) \times \operatorname{sum}(B\backslash C) \end{array}$$

Observe that the linear bijection ϕ always is continuous, but needs not to be open.

Proof Indication: Consider the vector subspaces $\operatorname{sum}(C)$ and $\operatorname{sum}(B\backslash C)$. Then define a map $\phi : \operatorname{sum}(C) \times \operatorname{sum}(B\backslash C) \to E$, $(x, y) \mapsto x + y$ from a product vector space into the original vector space. This map is linear because by the definition the relation $\phi(x, y) + s \cdot \phi(x', y') = \phi((x, y) + s \cdot (x', y'))$ is valid ($s \in \mathbb{K}$, $x, x' \in \operatorname{sum}(C)$, $y, y' \in \operatorname{sum}(B\backslash C)$). The injectivity follows from a kernel with a single element $\ker(\phi) = \{0\}$: The kernel consists of all those pairs (x, y), which sum the zero: $0 = x + y = \sum_{b \in C} x_b \cdot b + \sum_{b \in B\backslash C} y_b \cdot b$. Considering the new family $(z_b)_{b \in B}$, given by $z_b := \begin{cases} x_b & \text{if } b \in C, \\ y_b & \text{if } b \in B\backslash C, \end{cases}$ and combining this with a statement [AIII.41 §5.3 Prop. 3] about finite partitions of summable families results in $0 = \sum_{b \in B} z_b \cdot b$. The reduced freedom of B makes all scalars zero and asserts the injectivity of the map ϕ. Consider an element $v \in E$, due to the reduced basis, there exists a family $(v_b)_{b \in B} \in \mathbb{K}^B$ so that $v = \sum_{b \in B} v_b \cdot b$ the element is B-summable. The completeness of the vector space permits [TGIII.39 §5.3 Th. 2] to reassociate this linear sum to $v = \sum_{b \in C} v_b \cdot b + \sum_{b \in B\backslash C} v_b \cdot b$ giving an image of the map ϕ, thus making it surjective. Finally, define, using the isomorphisms φ_B, φ_C and $\varphi_{B\backslash C}$, an isomorphism $\varphi_B^{-1} \circ \phi \circ (\varphi_C \times \varphi_{B\backslash C})$ of the two coordinate spaces: $\mathbb{S}_B \cong \mathbb{S}_C \times \mathbb{S}_{B\backslash C}$. This establishes the commutative diagram. The map $\phi = + |_{\operatorname{sum}(C) \times \operatorname{sum}(B\backslash C)}$ is continuous because it is a restriction of a continuous function. ■

In those cases where the isomorphism ϕ can be shown to be open (or its inverse continuous), by definition, the direct sum becomes a topological direct sum.

38 **STATEMENT.** ***(Topologically Direct Sum)*** *If a topological vector space* E *is isomorphic to a direct sum* $E \cong_{\text{topological}} \operatorname{sum}(A) \oplus \operatorname{sum}(A')$ $(A, A' \subset E)$ *of linear sum subspaces,*

[15] For the notion "uniformity" see [TGII] and [TGIII.19 §3].

then those subspaces are closed $\mathrm{sum}\,(A) = \mathrm{cl}\,(\mathrm{sum}\,(A))$ *and the linear spans of the generating sets are dense* $\mathrm{sum}\,(A) = \mathrm{cl}\,(\mathrm{span}\,(A))$ *in the respective linear subspaces.*

Proof Indication: The statement [TGIII.47 §6.2 Remark 2] asserts that in topologically direct sums of subspaces the subspaces have to be closed. The rest follows from the Statement 20 on page 25 about the relations between the linear sum and the linear span. ■

Vector Space of Linear Combinations

Defining an algebraic basis means to introduce an isomorphism from a direct sum space onto the given vector space [AII.25 §1.11 Def. 10]. The converse is done in the following: Here, are used a given field and its direct sum space with respect to a given index set to construct a vector space.

A field $\mathbb{K}$ and a set B is everything needed to define a vector space: With respect to vector addition and scalar multiplication defined component-wise, the direct sum space $\mathbb{K}^{(B)} = \bigoplus_{b\in B} \mathbb{K}_b$ ($\mathbb{K}_b = \mathbb{K}$) is a vector space over the field $\mathbb{K}$. An algebraic basis is given by the set of all singly supported elements: $\left\{ (\delta_{bb'})_{b'\in B} \middle| b \in B \right\}$ (with the Kronecker delta $\delta_{bb'} := \begin{cases} 1 & \text{if } b = b' \\ 0 & \text{if } b \neq b' \end{cases}$). Elements of such a space are finitely supported families that can be written as linear combinations $x = \sum_{b\in B} x_b \cdot (\delta_{bb'})_{b'\in B}$. This kind of vector space which is given by sets of families or tuples (if the set B contains finitely many elements) is also called coordinate space. [AII.24 §1.11 first paragraph] The following definition makes it possible to clearly discern vectors b and their coordinates $(\delta_{bb'})_{b'\in B}$:

39 **DEFINITION** ***((Formal) Linear Combinations/Linear Span)*** *For a finitely supported family* $(x_b)_{b\in B} \in \mathbb{K}^{(B)}$ *of scalars consider the juxtaposition* $\sum_{b\in B} x_b \cdot b = x_{b_1} \cdot b_1 + \ldots + x_{b_n} \cdot b_n$ *that resembles a linear combination. This symbolic term is called* (formal) linear combination*. Then consider the set of formal linear combinations*

$$\mathrm{span}(B) = \mathrm{span}_{\mathbb{K}}(B) := \left\{ \textstyle\sum_{b\in B} x_b \cdot b \middle| (x_b)_{b\in B} \in \mathbb{K}^{(B)} \right\}$$

which is given the structure of a vector space by the component-wise definition of vector addition and scalar multiplication so that the map

$$\Phi : \mathbb{K}^{(B)} \to \mathrm{span}(B), \;\; (x_b)_{b\in B} \mapsto \textstyle\sum_{b\in B} x_b \cdot b$$

is an isomorphism of vector spaces. The vector space of those (formal) linear combinations is also called vector space of (formal) linear combinations[16] *or* linear span of set B*.*

The set B automatically is an algebraic basis of its linear span. Especially observe that this kind of linear span is not given as a subspace of any other vector space! The

[16] Often a vector-space of formal linear combinations $x = \sum_{b\in B} x_b \cdot b$ is considered as the coordinate space, where the singly supported elements $(\delta_{bb'})_{b'\in B}$ are identified with their indices b. [AII.25 §1.11 paragraph preceding Cor. 3]

bijection $\phi : \{ (\delta_{bb'})_{b' \in B} | \, b \in B \} \to B$, $(\delta_{bb'})_{b' \in B} \mapsto b$ suffices to describe the isomorphism Φ [AII.25 §1.11 Cor. 3]. This is based on the universal property of the coordinate space $\mathbb{K}^{(B)}$ and the inclusion map φ, which states that any function $f : \{ (\delta_{bb'})_{b' \in B} | \, b \in B \} \to E$ into a $\mathbb{K}$-vector space E can be uniquely extended to a $\mathbb{K}$-linear map $\overline{f} : \mathbb{K}^{(B)} \to E$ so that $f = \overline{f} \circ \varphi$ [AII.25 §1.11 Prop. 17]. In the case of a basis transformation the vector space $(E, +, \mathbb{K})$ remains the same, the coordinate spaces $\left(\mathbb{K}^{(B)}, +, \mathbb{K}\right)$ may change[17]; therefore a basis transformation is the identity on the original vector space, but a linear map between different coordinate spaces.

Vector Space of Linear Sums

For a given Hausdorff topological vector space the definition of a reduced basis involved an isomorphism from the subset of a product space onto the given vector space (see Definition 31 on page 33). Conversely, here is given a Hausdorff topological field, an index set and a topology on the product set of the indexed fields, then making a subset of the product space a Hausdorff topological vector space. From this is constructed an isomorphic independent Hausdorff topological vector space.

Consider a Hausdorff topological field $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$ and an index set B, then a subset $\mathcal{S}_B$ of the space $\mathbb{K}^B$ is supposed to be a Hausdorff topological vector space $(\mathcal{S}_B, +, \mathbb{K}, \mathcal{O}_{\mathcal{S}_B})$ with respect to addition and scalar multiplication defined component-wise having the set $\{ (\delta_{bb'})_{b' \in B} | \, b \in B \} \subset \mathcal{S}_B$ as a reduced basis.

40 **DEFINITION** ***(Formal Linear Sums/Linear Sum)*** *For a family $(x_b)_{b \in B} \in \mathbb{K}^B$ of scalars consider the notation $\sum_{b \in B} x_b \cdot b$ which resembles a linear sum. This symbolic term is called* (formal) linear sum. *For the Hausdorff topological vector space $(\mathcal{S}_B, +, \mathbb{K}, \mathcal{O}_{\mathcal{S}_B})$ consider the set of (formal) linear sums:*

$$\operatorname{sum}(B) = \operatorname{sum}_{\mathbb{K}}(B) := \left\{ \textstyle\sum_{b \in B} x_b \cdot b \,\middle|\, (x_b)_{b \in B} \in \mathcal{S}_B \right\}$$

which is given the structure of a vector space by the component-wise definition of vector addition and scalar multiplication so that the map

$$\Phi : \mathcal{S}_B \to \operatorname{sum}(B), \ \ (x_b)_{b \in B} \mapsto \textstyle\sum_{b \in B} x_b \cdot b$$

is an isomorphism of vector spaces. The image $\mathcal{O} := \Phi(\mathcal{O}_{\mathcal{S}_B})$ makes the range a Hausdorff topological vector space $(\operatorname{sum}_{\mathbb{K}, \mathcal{O}}(B), +, \mathbb{K}, \mathcal{O})$. This kind of space is called vector space of (formal) linear sums *or* linear $\mathcal{O}$-sum of set B *(That kind of vector space is not automatically embedded in any other vector space).*

The set B is a reduced basis of its linear sum (Statement 31 on page 33). If the direct sum subspace $\mathbb{K}^{(B)}$ is dense in $\mathcal{S}_B$, then the inclusion $\varphi : \{ (\delta_{bb'})_{b' \in B} | \, b \in B \} \to \operatorname{sum}(B)$, $(\delta_{bb'})_{b' \in B} \mapsto b$ suffices to describe the isomorphism Φ.

[17] The identification of singly supported elements and their indices, makes it difficult to describe basis transformations.

Text (ISBN 3-00-004696-8) to be found at http://www.schmarsow.de

Algebras of Monoids

This chapter analyzes associative unital algebras that are given by a commutative[18] field and a free monoid. The free monoid is the index set of the family of copies of the given field; the product set or the direct sum of that family can be given the structure of an associative unital algebra. The algebra, given by the direct sum space, is isomorphic to the tensor algebra. And the product space gives a large algebra. The last section introduces a sub-algebra of the algebra that is given by the product space: The algebra of formal power series, also called *series algebra*, is the Magnus-algebra which is given here a different topology, the so-called *product-box-topology*.

Monoidal Algebras

41 **NOTE. *(Monoidal Algebra)*** *Any monoid $(M,\cdot)$ [AI.12 §2.1 Def. 1] can be used as a basis for a vector space of linear combinations $\operatorname{span}_{\mathbb{K}}(M)$ over a commutative field $\mathbb{K}$ (Definition 39 on page 38). Due to the associativity and unitality of the monoid operation, the operation ($x_\alpha, y_\alpha \in \mathbb{K}$)*

$$\left(\sum_{\alpha\in M} x_\alpha\cdot\alpha\right)\cdot\left(\sum_{\beta\in M} y_\beta\cdot\beta\right) := \sum_{\gamma\in M}\left(\sum_{\gamma=\alpha\cdot\beta} x_\alpha\, y_\beta\right)\cdot\gamma$$

makes the linear span $\operatorname{span}(M)$ an associative unital algebra over the field $\mathbb{K}$ [AIII.19 §2.6 third sentence past formula (35)].

Such a kind of algebra has also a universal property [AIII.20 §2.6 Prop. 6]: Each morphism of (multiplicative) monoidal structures $f : M \to A$, where the set A has the structure of an associative unital $\mathbb{K}$-algebra and M is the given monoid, can be extended to a unique morphism of algebras $\overline{f} : \operatorname{span}(M) \to A$ so that $f = \overline{f}\circ\varphi$ (where the map $\varphi : M \to \operatorname{span}(M)$ is the inclusion).

Free Monoid

Consider the free monoid $(\operatorname{Mo}(X),\cdot)$ (The set X is called *generating set*.) [AI.78 §7.2] and the inclusion map $\varphi : X \to \operatorname{Mo}(X)$. The universal property states that the map $f : X \to$

[18] By the algebra definition [AIII.2 §1.1 Def. 1] part of the bilinearity of a non-trivial algebra multiplication requires the field to be commutative: Consider two algebra elements x, y and two scalars s, t, then follows that $(s\,x)\cdot(t\,y) = s(x\cdot(t\,y)) = s\,(t\,(x\cdot y)) = (s\,t)(x\cdot y)$ equals $(t\,s)(x\cdot y)$. If the algebra multiplication is non-trivial (so $x\cdot y \neq 0$), then $\mathbb{K}\,x\cdot y$ is a vector-space isomorphic to $\mathbb{K}$.

M into a monoid, can be extended to a unique morphism $\overline{f} : \mathrm{Mo}(X) \to M$ of monoids with $f = \overline{f} \circ \varphi$ [AI.79 §7.2 Prop. 3]. For the additively written monoid $(\mathbb{N}_0, +)$ consider the map $f : X \to \mathbb{N}_0$ which is given by $x \mapsto 1$. The morphic extension $\mathrm{L} : \mathrm{Mo}\,(X) \to \mathbb{N}_0$ of this map is also called word-length-logarithm, because the image values give the uniquely defined numbers of generating elements that make up the monoidal argument. The set $\{\mathrm{L}^{-1}(n) |\, n \in \mathbb{N}_0\}$ of inverse images of this map partitions the domain into sets of monoid elements that have the same word-length. These inverse images can be given a name: $\mathrm{Mo}_n\,(X) := \mathrm{L}^{-1}(n)$, $(n \in \mathbb{N}_0)$

42 **Statement.** ***(Finitely Many Reassociations)*** *Each element $\alpha \in \mathrm{Mo}\,(X)$ (X is a set) in a free monoid can be written as the product of only finitely many different pairs of elements: The set of factors of an element $F(\alpha) := \left\{ (\gamma, \delta) |\, (\gamma, \delta) \in \mathrm{Mo}\,(X)^2 \text{ and } \alpha = \gamma \cdot \delta \right\}$ contains finitely many ($\mathrm{L}(\alpha) + 1$) elements. [realizing AIII.27 §2.10 (D)]*

Proof Indication: Each element $\alpha \in \mathrm{Mo}\,(X)$ in a free monoid can be characterized by an n-tuple $(x_1, \ldots, x_n) \in X^n$ $(n \in \mathbb{N}_0)$ so that this element $\alpha = x_1 \cdot \ldots \cdot x_n$ is a product of generators [AI.79 §7.2 sentence before Prop. 2]. Since this characterization is unique and since the only element in free monoids, which is invertible, is the neutral element, monoid elements that can be multiplied to give the element α are subsequences of the n-tuple above, or the element α itself together with the neutral element: $F(\alpha) = \{(1, \alpha), (\alpha, 1)\} \cup \{ (x_1 \cdot \ldots \cdot x_k,\, x_{k+1} \cdot \ldots \cdot x_n) |\, 0 < k < n\}$ This set clearly has finitely many elements. Counting gives $n + 1$ pairs of elements. ■

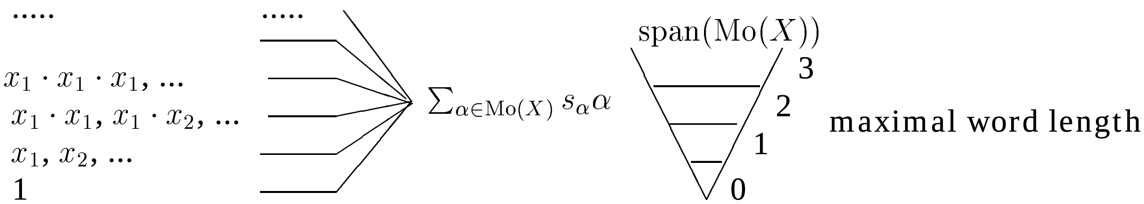
Direct Sum Space and Monoid: Tensor Algebra

Consider the associative unital algebra $(\mathrm{span}\,(\mathrm{Mo}\,(X))\,, +, \cdot, \mathbb{K})$ which is given by the set X (included naturally by $\varphi : X \to \mathrm{span}\,(\mathrm{Mo}\,(X))$ into the linear span of (formal) linear combinations) of generating elements of a free monoid and a commutative field $\mathbb{K}$ [AIII.21 §2.7 Def. 2]. A map $f : X \to A$ into an associative unital algebra $(A, +, \cdot, \mathbb{K})$ can be extended to a unique morphism $\overline{f} : \mathrm{span}(\mathrm{Mo}\,(X)) \to A$ of associative unital algebras so that $f = \overline{f} \circ \varphi$ (or $f(x) = \overline{f}(x)$ for all $x \in X$) [AIII.22 §2.7 Prop. 7]. This algebra is isomorphic to the tensor algebra of the vector space $\mathrm{span}_{\mathbb{K}}\,(X)$ [AIII.22 §2.7 Remark 1)] [AIII.62 §5.5 second sentence before the Remark].

$$\mathrm{span}\,(\mathrm{Mo}\,(X)) \cong_{\text{associative algebra}} \bigoplus_{n \in \mathbb{N}_0} (\mathrm{span}_{\mathbb{K}}\,(X))^{\otimes n}$$

The word-length-logarithm of the free monoid induces a direct sum decomposition (a graded direct sum) of that algebra with the associated order-function: Since inverse images of functions partition the domain, especially in the case of the word-length-logarithm, the associativity of direct sums [AII.12 §1.6 sentence before Cor. 1] makes the vector space $\mathrm{span}\,(\mathrm{Mo}\,(X))$ isomorphic to the direct sum $\bigoplus_{n \in \mathbb{N}_0} \mathrm{span}\,(\mathrm{Mo}_n\,(X))$, with generating sets $\mathrm{Mo}_n\,(X) := \{\alpha \,|\alpha \in \mathrm{Mo}\,(X) \text{ and } \mathrm{L}(\alpha) = n\}$. An associated filtration is given by the family of vector subspaces $\left(E_{(n)}\right)_{n \in \mathbb{N}_0}$, where $E_{(n)} = \bigoplus_{k=0}^{n} \mathrm{span}\,(\mathrm{Mo}_n\,(X))$.

The corresponding order-function $\mathrm{v} : \mathrm{span}(\mathrm{Mo}\,(X)) \to \mathbb{N}_0$ (see around Definition 24 on page 30) takes a linear combination of monoid elements and selects the maximal word-length of these.



Product Space and Free Monoid: A Large Algebra

Large monoidal algebras are introduced in [AIII.27 §2.10]. The condition (D) required there for the existence of such a large algebra is satisfied for free monoids in form of the Statement 42 above.

43 **NOTE.** ***(Coordinates of the Large Algebra of a Free Monoid)*** *Consider a set X and a commutative field $\mathbb{K}$. The product space $\mathbb{K}^{\mathrm{Mo}(X)}$ can be given an addition and scalar multiplication by defining these operations component-wise. The multiplication of two coordinate families $\underline{x} = (x_\alpha)_{\alpha\in\mathrm{Mo}(X)} \in \mathbb{K}^{\mathrm{Mo}(X)}$ and $\underline{y}$ (of possibly non-finite support) is defined by:*

$$\left(\underline{x}, \underline{y}\right) \mapsto \underline{x} \cdot \underline{y} := \left(\sum_{(\alpha,\beta)\in F(\gamma)} x_\alpha \, y_\beta \right)_{\gamma\in\mathrm{Mo}(X)}$$

Where $F(\gamma)$ ($\gamma \in \mathrm{Mo}\,(\mathrm{X})$) is the set of those pairs which have γ as their product (this summation condition can be rewritten simply as $\alpha \cdot \beta = \gamma$).

The algebra multiplication, as it is defined above, relies on the monoid multiplication and the (commutative) multiplication of the field. This makes the product space an associative unital algebra $\left(\mathbb{K}^{\mathrm{Mo}(X)}, +, \cdot, \mathbb{K}\right)$.

The definedness follows from the Statement 42. The idea for this definition comes from (formally) reassociating formal linear sums, like those used in the following large algebra. — If the field $(\mathbb{K}, \mathcal{O}_{\mathbb{K}})$ is Hausdorff topological and the product space $\mathbb{K}^B$ is given the product topology $\mathcal{O}_\pi$, then individual elements can be written as linear sums of possibly infinitely many singly supported elements (Statement 10 on page 18):

$$\underline{x} = \textstyle\sum_{\alpha\in\mathrm{Mo}(X)} \left(x_\alpha \cdot (\delta_{\alpha,\beta})_{\beta\in\mathrm{Mo}(X)} \right)$$

The Definition 40 (on page 39) of a linear sum gives the following:

44 **NOTE.** ***(Large Algebra of a Free Monoid)*** *The space $(\mathrm{sum}_{\mathbb{K},\mathcal{O}_\pi}(\mathrm{Mo}\,(X)), +, \cdot)$ which is given by the product algebra $\left(\mathbb{K}^{\mathrm{Mo}(X)}, +, \cdot, \mathbb{K}\right)$ is an isomorphic associative unital algebra. The free monoid $\mathrm{Mo}\,(X)$ is a reduced basis of the corresponding structure of a topological vector space. The algebra multiplication for elements $x = \sum_{\alpha\in\mathrm{Mo}(X)} x_\alpha \cdot \alpha$*

Text (ISBN 3-00-004696-8) to be found at http://www.schmarsow.de

and $y = \sum_{\beta\in\mathrm{Mo}(X)} y_\beta \cdot \beta$ *can be written as:*

$$x \cdot y := \sum_{\gamma\in\mathrm{Mo}(X)} \left(\sum_{(\alpha,\beta)\in F(\gamma)} x_\alpha \, y_\beta \right) \cdot \gamma$$

If the field is topologized by an absolute value, then this algebra is a topological algebra.

The structure of a topological vector space is immediate by the Example 33 (product) on page 33. Since the isomorphism of vector spaces identifies singly supported elements $(\delta_{\alpha,\beta})_{\beta\in\mathrm{Mo}(X)}$ and their support α and by the definition of the multiplications, the isomorphism is also that of associative unital algebras. — The continuity of a binary map $m:\ A_1 \times A_2 \to A'$, $(x_1, x_2) \mapsto x_1 \cdot x_2$ between topological spaces A_1, A_2, A' is shown by showing the continuity with respect to the product topology on $A_1 \times A_2$. Consider these three topological spaces to have the structure of a topological group or that of a topological vector space with the same field. If the binary map is a morphism, the criterion for continuity is given by an argument in one single point [TGIII.15 §2.8 Prop. 23], preferably the neutral element of the group operation. For commutative topological groups [TGIII.3 §1.2][19] where the binary map is a morphism in each of its arguments, the criterion for its continuity changes to

$$(\mathrm{B_I}) \quad \forall x_1, V \; (x_1 \in A_1 \,\text{and}\, V \in \mathcal{O}'(0) \Rightarrow \exists W_2 \; (W_2 \in \mathcal{O}_2(0) \,\text{and}\, x_1 \cdot W_2 \subset V))$$

$$(\mathrm{B_{II}}) \quad \forall V, x_2 \; (V \in \mathcal{O}'(0) \,\text{and}\, x_2 \in A_2 \Rightarrow \exists W_1 \; (W_1 \in \mathcal{O}_1(0) \,\text{and}\, W_1 \cdot x_2 \subset V))$$

$$(\mathrm{B_{III}}) \quad \forall V \; (V \in \mathcal{O}'(0) \Rightarrow \exists W_1, W_2 \; (W_1 \in \mathcal{O}_1(0) \,\text{and}\, W_2 \in \mathcal{O}_2(0) \,\text{and}\, W_1 \cdot W_2 \subset V))$$

If the binary map is a bilinear map $m : A \times A \to A'$ of algebras A and A' which are topological vector spaces $(A, +, \mathbb{K}, \mathcal{O})$ and $(A', +, \mathbb{K}, \mathcal{O}')$, the criteria [TGIII.48 §6.3 $(\mathrm{AV_I})$ and $(\mathrm{AV_{II}})$] can be formulated somewhat differently:

$$(\mathrm{AV'_I}) \quad \forall x, V \; (x \in A \,\text{and}\, V \in \mathcal{O}'(0) \Rightarrow \exists W \; (W \in \mathcal{O}(0) \,\text{and}\, x \cdot W \subset V \,\text{and}\, W \cdot x \subset V))$$

$$(\mathrm{AV'_{II}}) \quad \forall V \; (V \in \mathcal{O}'(0) \Rightarrow \exists W \; (W \in \mathcal{O}(0) \,\text{and}\, W \cdot W \subset V))$$

Where $\mathcal{O}(0)$ denotes the filter-base of open zero neighborhoods in the respective space with respect to the respective topology.

45 **STATEMENT.** ***(Continuity of Multiplication)*** *If the topology of the field is given by a non-trivial absolute value, then in the large algebra* $\left(\mathbb{K}^{\mathrm{Mo}(X)}, +, \cdot, \mathbb{K}, \mathcal{O}_\pi\right)$ *the multiplication*

$$\mathbb{K}^{\mathrm{Mo}(X)} \times \mathbb{K}^{\mathrm{Mo}(X)} \to \mathbb{K}^{\mathrm{Mo}(X)}; \qquad \left(\underline{x}, \underline{y}\right) \mapsto \underline{x} \cdot \underline{y}$$

is continuous with respect to the product topology.

Proof Indication: Since the multiplication can be written as a family of bilinear functions

[19] Here the case $A \times A \to A$ of a binary map (group operation) is treated for which all the spaces are equal.

$$m_\gamma : \mathbb{K}^{\mathrm{Mo}(X)} \times \mathbb{K}^{\mathrm{Mo}(X)} \to \mathbb{K}; \qquad \left(\underline{x}, \underline{y}\right) \mapsto \sum_{(\alpha,\beta)\in F(\gamma)} x_\alpha \, y_\beta \qquad \gamma \in \mathrm{Mo}\,(X)$$

into a space that has the product topology, the initial property of this topology [TGI.25 §4.1 Prop. 1] states that it is sufficient to show the continuity of every γ-indexed function above. Since the γ-indexed functions m_γ are bilinear, their continuity can be shown by verifying the conditions ($\mathrm{AV}'_{\mathrm{I}}$) and ($\mathrm{AV}'_{\mathrm{II}}$). The filter-base of open zero neighborhoods in the non-trivially valued field contains the disks $\mathrm{U}_\epsilon\,(0) = \{s \,|\, s \in \mathbb{K} \,\text{and}\, \mid s \mid < \epsilon\}$ with the positive real values $\epsilon > 0$. Furthermore define the set $F_\gamma := \mathrm{pr}_1\,(F(\gamma)) \cup \mathrm{pr}_2\,(F(\gamma))$ of all factors which can make up the monoid element $\gamma \in \mathrm{Mo}\,(X)$ (The set F_γ is finite because by Statement 42, the set $F(\gamma)$ is finite). Finally, let the natural number $n_\gamma := \mathrm{L}(\gamma)$ give the word-length of this monoid element.

($\mathrm{AV}'_{\mathrm{I}}$) Consider a family $\underline{x} = (x_\alpha)_{\alpha\in\mathrm{Mo}(X)} \in \mathbb{K}^{\mathrm{Mo}(X)}$ and a disk $\mathrm{U}_\epsilon\,(0) \subset \mathbb{K}$, then find an open zero neighborhood $W \in \mathcal{O}_\pi(0)$ so that $\underline{x} \cdot W \subset \mathrm{U}_\epsilon\,(0)$. — Of all those $\underline{x}$-coordinates that contribute to the γ-indexed function m_γ, fix the maximal absolute value $x_{\max} := \max\left(\{\mid x_\alpha \mid \,|\, \exists\beta\,((\alpha,\beta) \in F(\gamma))\}\right)$. With the previous information define the product set $V := \prod_{\beta\in\mathrm{Mo}(X)} V_\beta$ by setting $V_\beta := \begin{cases} \frac{1}{(n_\gamma+1)\cdot x_{\max}} \cdot \mathrm{U}_\epsilon\,(0) & \text{if } \beta \in F_\gamma, \\ \mathbb{K} & \text{if } \beta \in \mathrm{Mo}\,(X) \setminus F_\gamma. \end{cases}$ Since the set F_γ contains finitely many elements, the set V is an open zero-neighborhood in the product topology. Then for every element $\underline{v} \in V$ follows the inclusion $m_\gamma\,(\underline{x}, \underline{v}) = \sum_{(\alpha,\beta)\in F(\gamma)} x_\alpha \, v_\beta \subset \sum_{(\alpha,\beta)\in F(\gamma)} x_\alpha \cdot \frac{1}{(n_\gamma+1)\cdot x_{\max}} \cdot \mathrm{U}_\epsilon\,(0) \subset \mathrm{U}_\epsilon\,(0)$. The other case can be shown analogously.

($\mathrm{AV}'_{\mathrm{II}}$) Consider a disk $\mathrm{U}_\epsilon\,(0) \subset \mathbb{K}$ and the non-zero positive real number $\delta := \sqrt{\frac{\epsilon}{n_\gamma+1}}$. Define the product set $V := \prod_{\beta\in\mathrm{Mo}(X)} V_\beta$ by setting $V_\beta := \begin{cases} \mathrm{U}_\delta\,(0) & \text{if } \beta \in F_\gamma, \\ \mathbb{K} & \text{if } \beta \in \mathrm{Mo}\,(X) \setminus F_\gamma. \end{cases}$ Again, since the set F_γ contains finitely many elements, the set V is an open zero-neighborhood in the product topology. Consider two elements $\underline{v} = (v_\alpha)_{\alpha\in\mathrm{Mo}(X)}, \underline{v'} = \left(v'_\beta\right)_{\beta\in\mathrm{Mo}(X)} \in V$, then the γ-indexed function m_γ produces the following inclusion $m_\gamma\,(\underline{v}, \underline{v'}) = \sum_{(\alpha,\beta)\in F(\gamma)} v_\alpha \, v'_\beta \subset \left(\mathrm{U}_{(n_\gamma+1)\cdot\delta}\,(0)\right)^2 \subset \mathrm{U}_\epsilon\,(0)$, where the first inclusion uses the fact that the set $F(\gamma)$ contains $\mathrm{L}(\gamma) + 1$ pairs of monoid elements (Statement 42). ■

The Statement 7 on product topological closures of direct sums on page 15 justifies the following observation:

46 **NOTE.** ***(Density)*** *The tensor algebra (isomorphic to* $\mathrm{span}_{\mathbb{K}}\,(\mathrm{Mo}\,(X))$*) is dense within the product topologized large algebra of a free monoid (*$\mathrm{sum}_{\mathbb{K},\mathcal{O}_\pi}\,(\mathrm{Mo}\,(X))$*).*

This large (generally non-commutative) algebra is the set of all linear sums with non-commutative variables in the generating set X. But there is a special case:

47 **EXAMPLE.** ***(Algebra of all Power Series in a Single Variable)*** *If the generating set* $X = \{x\}$ *is a singleton, then the set* $\mathrm{sum}_{\mathbb{K},\mathcal{O}_\pi}\,(\mathrm{Mo}\,(\{x\}))$*, taken as a large algebra of a free monoid, is the algebra of all power series* $\sum_{n\in\mathbb{N}_0} s_n \cdot x^n$ *in a single variable (under the product topology of a topologically isomorphic product space* $\mathbb{K}^{\mathbb{N}_0}$*).*

This is immediate, since the any free monoid generated by a single element is isomorphic to the natural numbers: $\mathrm{Mo}\,(\{x\}) \cong \mathbb{N}_0$.

Product, Direct Sum and Free Monoid: Series Algebra

Linear sums $\sum_{\alpha\in\mathrm{Mo}(X)} s_\alpha\cdot\alpha$ in the large algebra of a free monoid are called *(formal) power-series in non-commuting variables* if the family of scalars $(s_\alpha)_{\alpha\in\mathrm{Mo}(X)}\in\mathbb{K}^{\mathrm{Mo}(X)}$ has those subfamilies $(s_\gamma)_{\gamma\in\mathrm{Mo}_n(X)}\in\mathbb{K}^{\mathrm{Mo}_n(X)}$ finitely supported which are indexed by monoid elements of the same word-length ($n\in\mathbb{N}_0$). The set of all those formal power-series in the linear sum $\mathrm{sum}_{\mathbb{K},\mathcal{O}_\pi}(\mathrm{Mo}(X))$ is isomorphic to the product of direct sum spaces $\prod_{n\in\mathbb{N}_0}\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$, where (also henceforth) $\mathbb{K}_\alpha=\mathbb{K}$. (Which can be seen as isomorphic to a subset of the coordinate space of the large algebra $\left(\mathbb{K}^{\mathrm{Mo}(X)},+,\cdot,\mathbb{K}\right)$)

48 **STATEMENT.** ***(Coordinates of the Series Algebra)*** *The subset $\mathcal{S}(X,\mathbb{K})\subset\mathbb{K}^{\mathrm{Mo}(X)}$ which is given by an isomorphism $\mathcal{S}(X,\mathbb{K})\cong\prod_{n\in\mathbb{N}_0}\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ is an associative unital subalgebra of the large coordinate algebra (Statement 43). If the generating set X contains finitely many elements, then the set $\mathcal{S}(X,\mathbb{K})$ equals the product space $\mathbb{K}^{\mathrm{Mo}(X)}$.*

Families like $\underline{x}=(x_\alpha)_{\alpha\in\mathrm{Mo}(X)}$ or $\underline{y}=(y_\beta)_{\beta\in\mathrm{Mo}(X)}$ inside $\mathcal{S}(X,\mathbb{K})\subset\mathbb{K}^{\mathrm{Mo}(X)}$ become doubly indexed $\underline{\underline{x}}=(\underline{x}_n)_{n\in\mathbb{N}_0}$ with finitely supported families $\underline{x}_n=(x_\alpha)_{\alpha\in\mathrm{Mo_n}(X)}$ inside the product space $\prod_{n\in\mathbb{N}_0}\mathbb{K}^{(\mathrm{Mo}_n(X))}$ of direct sums. In this space the multiplication can be rewritten to

$$\underline{\underline{x}}\cdot\underline{\underline{y}}:=\left(\sum_{n=j+k}\underline{x}_j\cdot\underline{y}_k\right)_{n\in\mathbb{N}_0}\quad\text{with}\;\underline{x}_j\cdot\underline{y}_k:=\left(\sum_{(\alpha,\beta)\in F(\gamma),\,\alpha\in\mathrm{Mo}_j(X),\,\beta\in\mathrm{Mo}_k(X)}x_\alpha\cdot y_\beta\right)_{\gamma\in\mathrm{Mo}_{j+k}(X)}$$

Proof Indication: The definedness of the multiplication within $\mathcal{S}(X,\mathbb{K})$ can be seen best in the form above: For all natural numbers $j,k\in\mathbb{N}_0$, the support $\mathrm{supp}\left(\underline{x}_j\cdot\underline{y}_k\right)$ is finite because each coordinate x_α or y_β makes only one contribution and the set $F(\gamma)$ contains only finitely many elements (Statement 42). — If the generating set X has finitely many elements so have the sets $\mathrm{Mo}_n(X)$ of monoid elements with word-length n ($n\in\mathbb{N}_0$). Finite index sets let direct sums degenerate into being product spaces: $\mathbb{K}^{(\mathrm{Mo}_n(X))}=\mathbb{K}^{\mathrm{Mo}_n(X)}$. The "associativity" of product spaces [AII.10 §1.5 sentence facing Prop. 5] shows the equality relation: $\mathcal{S}(X,\mathbb{K})=\mathbb{K}^{\mathrm{Mo}(X)}$ ■

Alternatively, the proof could be done by using a filtration topology and by identifying the series space with a completion with respect to that topology[20], but as can be seen on this level the use of any kind of topology is unnecessary.

The filtration topology mentioned before, needs a discretely topologized field. This can be avoided by considering a product-box-topology: Consider a Hausdorff topological commutative field $(\mathbb{K},\mathcal{O}_\mathbb{K})$. By the associativity of the products of spaces [EII.35 §5.5 Prop. 7] consider the isomorphy of sets: $\mathbb{K}^{\mathrm{Mo}(X)}\cong\prod_{n\in\mathbb{N}_0}\prod_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ Use the topology of the

[20] The family of two-sided ideals $\left(\bigoplus_{k=n}^{\infty}\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha\right)$ in the direct-sum algebra $\left(\mathbb{K}^{(\mathrm{Mo}(X))},+,\cdot,\mathbb{K}\right)$ is the filter-base of a topology that makes the algebra topological [TGIII.49 §6.3 Expl. 3]. If the direct-sum space is considered injected into the product space $\mathbb{K}^{\mathrm{Mo}(X)}$, then the completion of the direct-sum space with respect to that topology is given by $\mathcal{S}(X,\mathbb{K})\cong\prod_{n\in\mathbb{N}_0}\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$. The nature of the topology is that of a product topology of discretely topologized direct-sum spaces $\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ ($n\in\mathbb{N}_0$). Especially the field $\mathbb{K}$ has to be discretely topologized. [LieII.45 §5.1 paragraphs before Prop. 1] The emerging algebra is also called Magnus-algebra.

field to give the inner product spaces $\oplus_{\alpha\in \mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ ($n\in\mathbb{N}_0$) the box topology $\mathcal{O}^n_{\mathrm{box}}$ and then consider the outer product $\prod_{n\in\mathbb{N}_0}\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ as being product topologized. This topology henceforth is called *product-box-topology* $\mathcal{O}_{\mathrm{pb}}$ on the product space $\mathbb{K}^{\mathrm{Mo}(X)}$. The product-box-topology can be easily restricted $\mathcal{O}_{\mathrm{rpb}} := \mathcal{O}_{\mathrm{pb}}\cap \mathcal{S}(X,\mathbb{K})$ to the series algebra making it a topological space $(\mathcal{S}(X;\mathbb{K}),\mathcal{O}_{\mathrm{rpb}})$.

49 **STATEMENT.** ***(Series Algebra with Product-Box-Topology)*** *If the field $\mathbb{K}$ is commutative and has its topology given by an absolute value, then the restricted product-box-topology $\mathcal{O}_{\mathrm{rpb}}$ on the series algebra $\mathcal{S}(X,\mathbb{K})$ makes it a Hausdorff topological algebra. If the field is complete, then the topological series algebra is also complete.*

Proof Indication: Direct sum spaces $\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ ($n\in\mathbb{N}_0$) that are topologized by a restricted box topology are topological vector spaces by Statement[21] 14 on page 20.

(These direct sum spaces are closed by Statement 8 on page 16 and because the topological field is closed and having an absolute value, the field also has the topological Property T_1. According to [TGIII.73 Exercise §3.10] a box topologized product group is complete if all its factor groups, like a presupposedly complete field, are complete. With this and the fact that closed subgroups of complete groups remain complete [TGII.16 §3.4 Prop. 8], the direct sum spaces are complete.)

The product $\prod_{n\in\mathbb{N}_0}\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha$ of such topological vector spaces is also a topological vector space with respect to the product topology by Statement 9 on page 18. (Since the product uniformity induces the product topology [TGII.8 §2.3 Cor. of Prop. 4] and since complete factor uniformities make the product uniformity complete [TGII.17 §3.5 Prop. 10], the product space is also complete.) This makes the series algebra a (complete) Hausdorff topological vector space. — To show that the series algebra together with the restricted product-box-topology is a topological algebra it remains to show the continuity of the multiplication: Since the multiplication can be written as a family of functions

$$m_n:\mathcal{S}(X,\mathbb{K})\times\mathcal{S}(X,\mathbb{K})\to\oplus_{\alpha\in\mathrm{Mo}_n(X)}\mathbb{K}_\alpha;\qquad \left(\underline{x},\underline{y}\right)\mapsto \textstyle\sum_{n=j+k}\underline{x}_j\cdot\underline{y}_k\qquad n\in\mathbb{N}_0$$

into a space which is product topologized, the initial property of this topology [TGI.25 §4.1 Prop. 1] states that it is sufficient to show the continuity of every n-indexed function above. As shown before, the range of each function m_n is a topological group, therefore the addition used in the term $\sum_{n=j+k}\underline{x}_j\cdot\underline{y}_k$ is continuous and the proof can be restricted to considering the single summands $\underline{x}_j\cdot\underline{y}_k$:

$$m'_{(j,k)}:\mathcal{S}(X,\mathbb{K})\times\mathcal{S}(X,\mathbb{K})\to\oplus_{\alpha\in\mathrm{Mo}_{j+k}(X)}\mathbb{K}_\alpha;\qquad \left(\underline{x},\underline{y}\right)\mapsto\underline{x}_j\cdot\underline{y}_k\qquad (j,k)\in\mathbb{N}_0\times\mathbb{N}_0$$

Since for all sets $M\subset\mathrm{range}\left(m'_{(j,k)}\right)$ the inverse maps $m'^{-1}_{(j,k)}(M)$ are isomorphic to a set product $\prod_{\alpha\in\mathrm{Mo}(X)\setminus(\mathrm{Mo}_j(X)\cup\mathrm{Mo}_k(X))}\mathbb{K}_\alpha$ and a subset of $\oplus_{\alpha\in\mathrm{Mo}_j(X)\cup\mathrm{Mo}_k(X)}\mathbb{K}_\alpha$ due to the definition of the product topology, the maps $m'_{(j,k)}$ are continuous if the restrictions

$$m_{(j,k)}:\oplus_{\alpha\in\mathrm{Mo}_j(X)}\mathbb{K}_\alpha\times\oplus_{\alpha\in\mathrm{Mo}_k(X)}\mathbb{K}_\alpha\to\oplus_{\alpha\in\mathrm{Mo}_{j+k}(X)}\mathbb{K}_\alpha;\qquad\left(\underline{x}_j,\underline{y}_k\right)\mapsto\underline{x}_j\cdot\underline{y}_k\qquad (j,k)\in\mathbb{N}_0\times\mathbb{N}_0$$

are continuous with respect to the induced box topology.

The functions $m_{(j,k)}$ are bilinear maps between topological vector spaces, therefore show their continuity by verifying the conditions $(\mathrm{B}_{\mathrm{I}})$, $(\mathrm{B}_{\mathrm{II}})$ and $(\mathrm{B}_{\mathrm{III}})$ from page 44:

[21] The absolute value of the field is needed here.

(B_{I}) Consider an element $\underline{x}_j \in \oplus_{\alpha\in \mathrm{Mo}_j(X)}\mathbb{K}_\alpha$ and a $\mathrm{Mo}_{j+k}(X)$-family of positive real numbers $(\epsilon(\gamma))_{\gamma\in\mathrm{Mo}_{j+k}(X)}$, $\epsilon(\gamma) > 0$. Since each decomposition of a monoid element $\gamma \in \mathrm{Mo}_{j+k}(X)$ into two factors $\alpha \in \mathrm{Mo}_j(X)$ and $\beta \in \mathrm{Mo}_k(X)$ is unique, define a second $\mathrm{Mo}_k(X)$-family $(\delta(k,\beta))_{\beta\in\mathrm{Mo}_k(X)}$of non-zero positive real numbers by setting $\delta(k,\beta) := \begin{cases} \frac{\epsilon(\gamma)}{|x_\alpha|} & \text{if } \alpha \in \operatorname{supp}\left(\underline{x}_j\right), \\ \epsilon(\gamma) & \text{if } \alpha \in \mathrm{Mo}_j(X) \setminus \operatorname{supp}\left(\underline{x}_j\right). \end{cases}$ With the definitions $W_k := \oplus_{\beta\in\mathrm{Mo}_k(X)}\mathrm{U}_{\delta(k,\beta)}(0)$ and $V := \oplus_{\gamma\in\mathrm{Mo}_{j+k}(X)}\mathrm{U}_{\epsilon(\gamma)}(0)$, follow the inclusions $\underline{x}_j \cdot W_k \subset \oplus_{\beta\in\mathrm{Mo}_k(X)} |x_\alpha| \cdot \mathrm{U}_{\delta(k,\beta)}(0) \subset V$.

(B_{II}) This can be done analogously.

(B_{III}) For given natural numbers $j,k \in \mathbb{N}_0$ consider a family of positive real numbers $(\epsilon(\gamma))_{\gamma\in\mathrm{Mo}_{j+k}(X)}$, $\epsilon(\gamma) > 0$. Each of these indices can be written as a product of exactly two monoid elements with the respective word-lengths j and k. With this information define two families $(\delta(j,\alpha))_{\alpha\in\mathrm{Mo}_j(X)}$ and $(\delta(k,\beta))_{\beta\in\mathrm{Mo}_k(X)}$ of non-zero positive real numbers by setting $\delta(j,\alpha) = \delta(k,\beta) := \sqrt{\epsilon(\gamma)}$. With that consider the boxes $W_j := \oplus_{\alpha\in\mathrm{Mo}_j(X)}\mathrm{U}_{\delta(j,\alpha)}(0)$ and $W_k := \oplus_{\beta\in\mathrm{Mo}_k(X)}\mathrm{U}_{\delta(k,\beta)}(0)$ and the set $W_j \cdot W_k$ of all the products of their elements. The inclusion relation $W_j \cdot W_k \subset \prod_{\gamma\in\mathrm{Mo}_{j+k}(X)} \sum_{(\alpha,\beta)\in F(\gamma)} \mathrm{U}_{\delta(j,\alpha)}(0) \cdot \mathrm{U}_{\delta(k,\beta)}(0)$ follows immediately. Due to the restrictions the sum over the set $F(\gamma)$ is trivial, and due to the relation $\mathrm{U}_{\delta_1}(0) \cdot \mathrm{U}_{\delta_2}(0) \subset \mathrm{U}_{\delta_1\cdot\delta_2}(0)$ inside the field, follows $W_j \cdot W_k \subset \prod_{\gamma\in\mathrm{Mo}_{j+k}(X)} \mathrm{U}_{\epsilon(\gamma)}(0)$. Due to the definedness of the multiplication finally follows $W_j \cdot W_k \subset V := \oplus_{\gamma\in\mathrm{Mo}_{j+k}(X)}\mathrm{U}_{\epsilon(\gamma)}(0)$. ■

Note that since the direct sum space $\mathbb{K}^{(\mathrm{Mo}(X))}$ is a subspace of the product space $\mathbb{K}^{\mathrm{Mo}(X)}$ and the following inclusion is valid:

$$\mathbb{K}^{(\mathrm{Mo}(X))} \subset \mathcal{S}(X,\mathbb{K}) \subset \mathbb{K}^{\mathrm{Mo}(X)}$$

Ideals and Envelopes in Algebras

There are some standard ways to formulate a multiplication operation on a vector space with a commutative field:

- A vector space with an algebraic basis can be given a bilinear operation by using the universality of the tensor product [AIII.10 §1.7].
- If the algebraic basis of a vector space is a monoid (or a magma [AI.1 §1.1 Def. 1]), then the respective operation gives a multiplication table [AIII.19 §2.6].
- For a given algebra enrich the existent multiplication by a quotient process [AIII.3 §1.2 paragraph about quotient algebras].

Can quotient processes on the large algebras of the previous chapters introduce commutator relations, like $a \cdot b - b \cdot a = 1$? Therefore consider various methods to partition the existing algebras with ideals [AI.98 §8.6 Def. 4] [AIII.4 §1.2 paragraph about ideals] and their cosets so that the multiplication remains defined.

Ideals and Elementary Envelopes

Any subset $I \subset M$ in a monoid $(M, \cdot)$ satisfying the inclusion relations $I \cdot M \subset I$ and $M \cdot I \subset I$ is called a *two-sided (semigroup) ideal*. The monoid M together with a commutative field $\mathbb{K}$ can be used to construct a monoidal algebra $A := \mathrm{span}_{\mathbb{K}}(M)$ (Note 41 on page 41). Ideals $I' \subset A_{\mathbb{K}}$ in this associative unital algebra are vector subspaces satisfying relations $I' \cdot A \subset I'$ and $A \cdot I' \subset I'$ which constitute the notion of *multiplicative stability*. A multiplicatively stable set $S \subset A$ inside the algebra makes the linear span $\mathrm{span}(S)$ an ideal of that algebra [AI.99 §8.6 Prop. 3 extended]. A special case of a multiplicatively stable set is the two-sided semigroup ideal $I \subset M$ which is injected into the algebra. Another special case is the set $\epsilon(R, F) := \{R, FR, RF, (FR)F, F(RF), (RF)F, \ldots\}$ (the set $R \subset A$ is called the set of *relators* or also *generators of the ideal*), for $F = A$ that is just the set of all multiplicative envelopes of the relators, if the set F is an algebraic basis of the algebra, that kind of set is called the *set of all elementary envelopes*. Often the construction of an ideal is abbreviated by writing

$$\langle R \rangle_{\mathrm{ideal}} := \mathrm{span}(\epsilon(R, A))$$

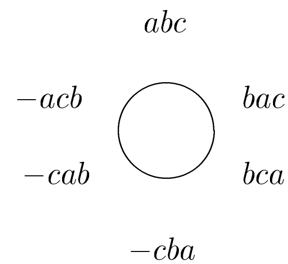
.

50 **EXAMPLE.** ***(Linearly Dependent Elementary Envelopes)*** *Consider a commutative field* $\mathbb{K}$*, a totally ordered set* $X = \{a, b, c\}$ *(*$a \leq b \leq c$*), generating a free monoid* $\mathrm{Mo}(X)$ *and the emerging spanned algebra* $\mathrm{span}_{\mathbb{K}}(\mathrm{Mo}(X))$*.*

In this algebra consider commutator elements $ab - ba$*,* $ac - ca$ *and* $bc - cb$ *which are linearly independent. Multiplication with monoid generators already produces linearly dependent elements (Jacobi-Identity):*

$$\begin{aligned} &(abc - bac) + (bac - bca) + (bca - cba) - \\ &-(-cba + cab) - (-cab + acb) - (-acb + abc) \quad = 0 \end{aligned}$$

So, generally elementary envelopes are linearly dependent.



— The notion of an ideal also can be described by a linear sum in a topological algebra:

51 **STATEMENT.** ***(Linear Sum as an Ideal)*** *Consider a complete Hausdorff topological algebra* $(A, +, \cdot, \mathcal{O}, \mathbb{K})$ *over a complete Hausdorff topological commutative field* $\mathbb{K}$*. Therein consider a subset* $R \subset A$ *(a set of relators). The linear sum* $\mathrm{sum}_{\mathcal{O}}(\epsilon(R, A)) \subset A$ *of the multiplicative envelopes of the relators is a (two-sided) ideal of the algebra.*

Proof Indication: By Statement 17 on page 24 the linear sum is a vector space. It remains to show the multiplicative stability of the linear sum: Consider an element $a \in A$ and an element of the linear sum given by an $\epsilon(R, A)$-summable family of scalars $(s_e)_{e \in \epsilon(R,A)} \in \mathbb{K}^{\epsilon(R,A)}$. Due to the continuity of the algebra multiplication, the product $a \cdot \sum_{e \in \epsilon(R,A)} s_e\, e$ can be rewritten as a linear sum $\sum_{e \in \epsilon(R,A)} s_e\, a \cdot e$. Since the set of multiplicative envelopes is stable under algebra multiplication, the products $a \cdot e \in \epsilon(R, A)$ are inside the set of multiplicative envelopes. The map $m : \epsilon(R, A) \to \epsilon(R, A)$, given by $e \mapsto a \cdot e$, lets its inverse images partition the domain, with the partition sets $P_{e'} := m^{-1}(e')$. The completeness of the algebra allows the following reassociation [TGIII.39 §5.3 Th. 2] $\sum_{e \in \epsilon(R,A)} s_e\, a \cdot e = \sum_{e' \in \epsilon(R,A)} \sum_{e \in P_{e'}} s_e\, a \cdot e$, where $a \cdot e = e'$, giving the term $\sum_{e' \in \epsilon(R,A)} \left(\sum_{e \in P_{e'}} s_e \right) e' \in \mathrm{sum}_{\mathcal{O}}(\epsilon(R, A))$. The case of the commuted product is shown analogously. ∎

A Degenerate Case

52 **EXAMPLE.** ***(Trivial Ideals in Large Algebras)*** *Consider a commutative field* $\mathbb{K}$ *which is topological and complete. For a generating set* $X = \{x\}$ *consider the large topological algebra* $A_{\mathbb{K}} := \mathrm{sum}_{\mathbb{K}, \mathcal{O}_\pi}(\mathrm{Mo}(X))$ *of the corresponding free monoid* $\mathrm{Mo}(X)$*. Then define*

the relator $r := 1 + x \cdot x$*; and note that the linear sum* $-\sum_{n\in\mathbb{N}} r \cdot (-x)^{2(n-1)} = 1$ *is equal to one. So the ideal* $\mathrm{sum}_{\mathbb{K}}\left(\epsilon(\{r\}, A_{\mathbb{K}})\right)$ *contains the multiplicative neutral element and therefore equals the entire algebra.*

Analogously such an argument can be given for the Heisenberg relator $a \cdot b - b \cdot a - 1$ *in an algebra* $\mathrm{sum}_{\mathbb{K},\mathcal{O}_\pi}\left(\mathrm{Mo}\left(X\right)\right)$ *with a generating set* $X = \{a, b\}$*.*

This shows that the product topology is too coarse to allow for such summed ideals to be non-trivially topologically supplemented.

Monoidal Quotient Algebras

If a given algebra is partitioned by the cosets of an ideal, then the partition can be given the structure of an algebra, the so-called quotient algebra [AIII.4 §1.2]. If the ideal is given by a set of generators, also called *relators*, then the quotient algebra is also called *enveloping algebra*[22]. The ideal is the additive zero in that algebra and elements of this ideal represent that zero element; so especially the generators of the ideal are also zero and therefore represent defining relations of the quotient- or enveloping algebra.

Here, primarily are considered associative unital algebras with commutator relations. Such algebras can be characterized by non-associative algebras called Lie algebras, with the antisymmetry and the Jacobi-Identity of the multiplication as their defining properties [AIII.24 §2.8 Expl. 1)][LIEI.9 §1.2 Def. 1]. For Lie algebras $\mathbf{g}$ with an algebraic basis of finitely many elements – the reverse process – the construction of an associative unital algebra with the proper commutation relations, from a given Lie algebra, is established by the Poincaré-Birkhoff-Witt Theorem [LIEI.30 §2.7]. The associative unital algebra constructed there is called *universal enveloping algebra* $\mathrm{U}(\mathbf{g})$ *of the Lie algebra* $\mathbf{g}$.

First in this chapter is shown that there exist other associative unital algebras than the universal enveloping algebra of a Lie algebra which characterize a special kind of Lie algebras, namely those with non-zero center.

Such structures are given by the Heisenberg-Lie algebra and the Heisenberg associative unital algebra, as well as a corresponding pair of oscillator algebras. These are presented in the way they naturally occur in the context of physics and the multiplication operations are made explicit.

Centered Enveloping Algebra

First consider a totally ordered set $(X, \leq)$ (see footnote on page 29) generating a free monoid $(\mathrm{Mo}\,(X)\,, \cdot, \leq)$ with a lexicographical order on its elements [EIII.22 §2.6]. A monoid element $x = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ ($x_i \in X$ and $n \in \mathbb{N}_0$) is called *normally ordered* if and only if the family $(x_i)_{i\in\{1,\ldots,n\}}$ of generators is of the ascending kind :

$$\forall i, j\, (i, j \in \{1, \ldots, n\} \,\text{and}\, i \leq j \Rightarrow x_i \leq x_j)$$

Then consider the subset of all normally ordered monoid elements in the free monoid

$$\mathrm{PBW}\,(X) := \{x\,|x \in \mathrm{Mo}\,(X) \;\text{and}\, x \,\text{is normally ordered}\}$$

[22] Because, in constructing the (two-sided) ideal explicitly, the generating elements are enveloped by multiplying algebra elements from both sides. (Enveloping algebras of Lie algebras, as defined in [LIEI.22 §2.1 Def. 1] or [DIX.69 2.1.1] are special enveloping algebras.)

and call it the *Poincaré-Birkhoff-Witt set*.

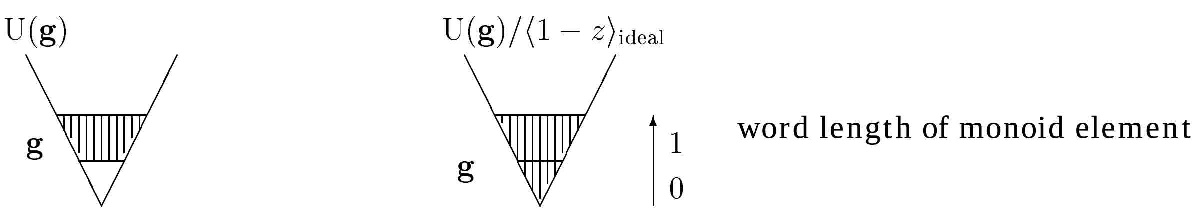
Furthermore consider a Lie algebra $(\mathbf{g}, +, [.], \mathbb{K})$ over one of the commutative fields $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (the real and complex numbers) with the (non-associative) multiplication $[.] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ satisfying the antisymmetry relation and the Jacobi-Identity. Let this Lie algebra have a totally ordered algebraic basis $(B, \leq)$ with $B \subset \mathbf{g}$, so that the space is the linear span $\mathbf{g} = \mathrm{span}_{\mathbb{K}}(B)$ of the algebraic basis.

The vector space structure $(\mathrm{span}_{\mathbb{K}}(B), +, \mathbb{K})$ of this Lie algebra can be injected easily into the associative unital algebra $A := \mathrm{span}_{\mathbb{K}}(\mathrm{Mo}(B))$ when defining a linear map $i : \mathbf{g} \to A$ by $b \mapsto b\,(b \in B)$. The associative unital algebra $\mathrm{U}(\mathbf{g}) := A / \langle \{R(x, y) \mid x, y \in g\} \rangle_{\mathrm{ideal}}$, which is given by the quotient of the algebra A and the ideal generated by the relators $R(x, y) := i(x) \cdot i(y) - i(y) \cdot i(x) - i([x, y])$ $(x, y \in \mathbf{g})$, satisfies commutation relations with those results that are given by the bracket operations of the Lie algebra. Now, for finite dimensional Lie algebras $\mathbf{g}$ (the algebraic basis B contains finitely many elements), the Poincaré-Birkhoff-Witt Theorem states:

- the quotient algebra $\mathrm{U}(\mathbf{g})$ is isomorphic to $\mathrm{span}_{\mathbb{K}}(\mathrm{PBW}(B))$
- the morphism σ of Lie algebras $\sigma : \mathbf{g} \to \mathrm{U}(\mathbf{g})$, which transforms bracket relations $\sigma([x, y]) = \sigma(x) \cdot \sigma(y) - \sigma(y) \cdot \sigma(x)$ $(x, y \in \mathbf{g})$ into commutator relations of the associative unital algebra $\mathrm{U}(\mathbf{g})$, is injective.

For Lie algebras with a non-empty center[23], the following statement changes this statement minimally:

53 **STATEMENT.** ***(Centered Enveloping Algebra)*** *Consider a finite-dimensional Lie algebra* $(\mathbf{g}, +, [.], \mathbb{K})$ *with a totally ordered algebraic basis* $(B, \leq)$ *and its universal enveloping algebra as described above. Let there be an element* z *of the algebraic basis* B *in the center of the Lie algebra* $z \in B \cap Z(\mathbf{g})$ *and define the set* $X := B \backslash \{z\}$*. Then the map* $\tau_z : \mathbf{g} \to \mathrm{U}(\mathbf{g}) / \langle 1 - z \rangle_{\mathrm{ideal}}$*, given by* $f : B \to \mathrm{Mo}(X)$, $b \mapsto c := \begin{cases} b & \text{if } b \in B \backslash \{z\}, \\ 1 & \text{if } b = z, \end{cases}$ *is an injective morphism of Lie algebras. The image* $\tau_z(\mathbf{g})$ *of the map is isomorphic to* $\mathrm{span}(\{1\} \cup X)$ *and the range of the map is isomorphic to* $\mathrm{span}(\mathrm{PBW}(X))$.



Also alluding to the central character of the element z the new algebra is called *centered enveloping algebra* $C_z(\mathbf{g})$ *of the Lie algebra* $\mathbf{g}$. (For special cases of Heisenberg-like algebras such an algebra is also called Weyl-algebra [DIX.144 4.6.3].)

[23] Actually an element z in any kind of magma $(M, *)$ is called a *central element* if it commutes with all other elements $\forall m\,(m \in M \Rightarrow z * m = m * z)$. The set of all those elements is called "center" $Z(M) = \{z \,|\, z \in M \text{ and } \forall m\,(m \in M \Rightarrow z * m = m * z)\}$ [AI.8 §1.5 Def. 10]. Since the operation of Lie-algebras can be understood as using a bracket for coding the commutator of an associative algebra the *center of a Lie-algebra* $(g, +, [.], \mathbb{K})$ is defined differently: $Z(g) := \{z \,|\, z \in g \text{ and } \forall x\,(x \in g \Rightarrow [z, x] = 0)\}$[LIEI.15 §1.6 sentence before the last]!

Proof Indication: Step 1) The map f inflates by the universal property of the free monoid [AI.79 §7.2 Prop. 3] to a morphism $f_1 : \mathrm{Mo}\,(B) \to \mathrm{Mo}\,(X)$ of monoids. Since the set B is totally ordered, there exists the set of normally ordered elements $\mathrm{PBW}\,(B) \subset \mathrm{Mo}\,(B)$, to which this map can be restricted: $f_2 := f_1 \Big|_{\mathrm{PBW}(B)}$ The definition of the map f makes f_2 the map which just omits any z-generator, from that follows the inclusion relation $f_2\,(\mathrm{PBW}\,(B)) \subset \mathrm{PBW}\,(X)$. The corestriction $f_3 : \mathrm{PBW}\,(B) \to \mathrm{PBW}\,(X)$ can be extended by the universal property of vector spaces [AII.25 §1.11 Cor. 3] to a linear function $f_4 : \mathrm{span}_{\mathbb{K}}\,(\mathrm{PBW}\,(B)) \to \mathrm{span}_{\mathbb{K}}\,(\mathrm{PBW}\,(X))$ with $\sum_{\beta \in \mathrm{PBW}(B)} s_\beta \cdot \beta \mapsto \sum_{\alpha \in \mathrm{PBW}(X)} t_\alpha \cdot \alpha$; where the scalars $t_\alpha := \sum_{\beta \in f_3^{-1}(\alpha)} s_\beta$ are sums of subfamilies of the family $(s_\beta)_{\beta \in \mathrm{PBW}(B)} \in \mathbb{K}^{(\mathrm{PBW}(B))}$. (The set $f_3^{-1}(\alpha)$ gathers all those monoid elements which become α if the generator z is omitted.)

Step 2) The Poincaré-Birkhoff-Witt Theorem gives an isomorphism $\Phi : \mathrm{span}\,(\mathrm{PBW}\,(B)) \to \mathrm{U}(\mathbf{g})$ of associative unital algebras with $\sum_{\beta \in \mathrm{PBW}(B)} s_\beta \cdot \beta \mapsto s$ for families $(s_\beta)_{\beta \in \mathrm{PBW}(B)} \in \mathbb{K}^{(\mathrm{PBW}(B))}$ of scalars. The quotient map $q : \mathrm{U}(\mathbf{g}) \to \mathrm{U}(\mathbf{g}) / \langle 1 - z \rangle_{\mathrm{ideal}}$ with $s \mapsto s + \langle 1 - z \rangle_{\mathrm{ideal}}$ is a surjective morphism of associative unital algebras. Finally define a map $\Psi : \mathrm{span}\,(\mathrm{PBW}\,(X)) \to \mathrm{U}(\mathbf{g}) / \langle 1 - z \rangle_{\mathrm{ideal}}$ by assigning a linear combination to a coset $\sum_{\alpha \in \mathrm{PBW}(X)} t_\alpha \cdot \alpha \mapsto \Phi\left(\sum_{\alpha \in \mathrm{PBW}(X)} t_\alpha \cdot \alpha\right) + \langle 1 - z \rangle_{\mathrm{ideal}}$. With the criterion

$$\Phi\left(\sum_{\beta \in \mathrm{PBW}(B)} s_\beta \cdot \beta\right) \in \langle 1 - z \rangle_{\mathrm{ideal}} \Leftrightarrow \forall \alpha \left(\alpha \in \mathrm{PBW}\,(X) \Rightarrow 0 = \sum_{\beta \in f_3^{-1}(\alpha)} s_\beta\right) \qquad (*)$$

and the previous definitions and conventions the following relation can be shown elementwise $\Psi \circ f = q \circ \Phi$. The left side of this relation, as a composition of surjective functions, makes the map Ψ surjective. Since the map Ψ is linear, consider the kernel $\ker(\Psi) = \{t \,|t \in \mathrm{span}\,(\mathrm{PBW}\,(X)) \text{ and } \Psi\,(t) = \langle 1 - z \rangle_{\mathrm{ideal}}\}$; an element of this kernel is of the form $\Phi\left(\sum_{\alpha \in \mathrm{PBW}(X)} t_\alpha \cdot \alpha\right) \in \langle 1 - z \rangle_{\mathrm{ideal}}$ which by the Criterion $(*)$ makes the kernel $\ker(\Psi) = \{0\}$ a singleton. So the map Ψ is a isomorphism of vector spaces.

$$\begin{array}{ccc} \mathrm{span}(\mathrm{PBW}(B)) & \xrightarrow{\ \Phi\ } & \mathrm{U}(\mathbf{g}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{span}(\mathrm{PBW}(X)) & \xrightarrow{\ \Psi\ } & \mathrm{U}(\mathbf{g})/\langle 1 - z \rangle_{\mathrm{ideal}} \end{array}$$

Step 3) Since the restricted composition $(\Psi \circ f \circ \Phi^{-1} \circ \sigma)\,|_B\,(B) = X \cup \{1\}$ conserves the cardinality of the algebraic basis B and produces a set of linearly independent elements, the map $q \circ \sigma$ is an injective morphism of vector spaces. And since the quotient map q is a morphism of associative algebras, it follows that the composition $q \circ \sigma$ is a morphism of Lie algebras. ■

Note that for the associative quotient algebras considered so far in this chapter ($\mathrm{U}(\mathbf{g})$ and $\mathrm{U}(\mathbf{g}) / \langle 1 - z \rangle_{\mathrm{ideal}}$), the filtration of maximal word-length (as indicated on page 43) remains compatible with the multiplication of the algebra (as indicated in [LIEII.38 §4.1 last paragraph]), but in these cases, where the relators contain monoid elements of different word-length (see the case of the ideal $\langle 1 - z \rangle_{\mathrm{ideal}}$), the associated direct sum space is not compatible with the algebra multiplication.

Heisenberg Algebra: Lie and Associative

The three-dimensional complex vector space $\mathbf{h} := \mathrm{span}_{\mathbb{C}}(\{1, a, b\})$ together with an anti-symmetric bracket multiplication $[,] : \mathbf{h} \times \mathbf{h} \to \mathbf{h}$, given by $[b, a] := 1$, becomes a complex Lie algebra, the so-called *Heisenberg Lie algebra*. Then consider the set $\{a, b\}$ with the total order $a \leq b$. The centered enveloping algebra $C_1(\mathbf{h})$ of this Lie algebra (The element 1 of the Lie algebra is associated with the neutral element of the free monoid.) is also called *associative Heisenberg algebra*. This algebra is given by the linear span $\mathrm{span}_{\mathbb{C}}(\mathrm{PBW}(\{a, b\}))$ which is generated by the set $\mathrm{PBW}(\{a, b\}) = \left\{ a^k\, b^l \,\middle|\, k, l \in \mathbb{N}_0 \right\}$ of normally ordered monoid elements and where the associative multiplication satisfies the Heisenberg commutation relation $b\,a = a\,b + 1$. This relation transforms *non*-normally ordered products of elements a and b into normally ordered products.

54 **NOTE.** ***(Heisenberg Commutation Relation Iterated)*** *Defining $b^{-1} := 0$ in the associative Heisenberg algebra $C_1(\mathbf{h})$, the iteration of the Heisenberg commutation relation $b\,a - a\,b = 1$ yields ($s, t \in \mathbb{N}_0$ and $s! := s \cdot (s-1) \cdot \ldots \cdot 1$):*

$$b^s\, a^t = t!\, s! \sum_{m=0}^{\min(t,s)} \frac{a^{t-m}\, b^{s-m}}{(t-m)!\, m!\, (s-m)!}$$

Known special cases that are used in the proof by induction are $b\,a^k = k\,a^{k-1} + a^k\,b$ and $b^l\,a = l\,b^{l-1} + a\,b^l$ $(k, l \in \mathbb{N}_0)$. Then consider the following change of summation indices:

55 **STATEMENT.** ***(Summation Index Change)*** *Consider a complete topological group $(G, +, \mathcal{O})$ which is commutative. And a family $S : \mathbb{Z}^5 \to G$ ($\mathbb{Z}$ denotes the module of integers.) for which one of the multiple sums below exists. Then the following recommutation is valid:*

$$\sum_{r,s,t,u \in \mathbb{N}_0} \sum_{m=0}^{\min(s,t)} S(r, s, t, u, m) = \sum_{k,l,n \in \mathbb{N}_0} \sum_{q=0}^{k} \sum_{p=0}^{l} S(k-q, n+p, n+q, l-p, n)$$

Proof Indication: The relations $r = k - q$, $s = n + p$, $t = n + q$, $u = l - p$ and $m = n$ define an automorphism on the module $\mathbb{Z}^5$. The restrictions on the summation indices define sets, that are mapped onto each other. ■

Especially in those cases where the family S has only finitely many non-trivial values and the group is just a commutative group, the equality above remains true.

The previous observations allow to explicitly formulate the multiplication of the associative Heisenberg algebra:

56 **NOTE.** ***(Associative Heisenberg Multiplication)*** *Consider two elements $A, B \in C_1(\mathbf{h})$ which are given by families $(A_{r,s})_{r,s \in \mathbb{N}_0}, (B_{t,u})_{t,u \in \mathbb{N}_0} \in \mathbb{C}^{(\mathbb{N}_0 \times \mathbb{N}_0)}$ of scalars and the associated linear combinations $A = \sum_{r,s \in \mathbb{N}_0} A_{r,s}\, a^r\, b^s$ and $B = \sum_{t,u \in \mathbb{N}_0} B_{t,u}\, a^t\, b^u$. Then the algebra multiplication takes the following form:*

$$(A, B) \mapsto A\,B = \sum_{k.l \in \mathbb{N}_0} \left(\sum_{n \in \mathbb{N}_0} \sum_{q=0}^{k} \sum_{p=0}^{l} \frac{(n+p)!\,(n+q)!}{p!\,n!\,q!} A_{k-q,n+p}\, B_{n+q,l-p} \right) a^k\, b^l$$

This multiplication cannot be extended easily to linear sums which can be seen in the following example:

57 **EXAMPLE.** ***(Extensibility Problem)*** *Inside the Hausdorff topological vector space* $\mathrm{sum}_{\mathbb{C},\mathcal{O}_\pi}(\mathrm{PBW}(\{a,b\})) \cong \mathbb{C}^{\mathrm{PBW}(\{a,b\})}$ *consider the element X which is given by the linear sum $\sum_{r,s\in\mathbb{N}_0} 1\, a^r\, b^s$. If a multiplication of this element to itself X^2 is attempted using the formula above, then the resultant coordinate for the monoid element 1 is undefined, giving the problematic term $\sum_{n\in\mathbb{N}_0} n!$.*

Oscillator Algebra

The term "oscillator" arises from the physics of the quantum mechanical harmonic oscillator ([BÖHM Chap. II][FICK Part 4. Chap. 2][SAKU Chap. 2.3]). There, observables like position X, linear momentum P and energy H are described by non-commuting symbols: $H = \frac{1}{2m}P^2 + \frac{k}{2}X^2$ with $\omega = \sqrt{\frac{k}{m}}$ and the commutation relation $P\,X - X\,P = -i\hbar 1$ (m is the mass term, k describes the intensity of the quadratic potential, ω is the characteristic frequency of the oscillator, $\hbar$ is Planck's constant divided by 2π and i denotes the imaginary unit of the complex numbers.) A transformation introduces the observables in natural units: $p := \frac{1}{\sqrt{\hbar m\omega}}P$, $x := \sqrt{\frac{m\omega}{\hbar}}X$ and $h := \frac{H}{\hbar\omega}$ with the associated relations $h = \frac{p^2+x^2}{2}$ and $p\,x - x\,p = -i1$. A linear transformation (*) which is defined by $b := \frac{1}{\sqrt{2}}(x+ip)$ and $a := \frac{1}{\sqrt{2}}(x-ip)$ gives the notation that is used henceforth:

$$h = a\,b + \tfrac{1}{2} \text{ and } b\,a - a\,b = 1 \text{ with}$$
$$h\,a - a\,h = a \text{ and } h\,b - b\,h = -b$$

These heuristics make it possible to introduce the so-called *oscillator Lie algebra* as a four-dimensional complex vector space $\mathbf{o} := \mathrm{span}_{\mathbb{C}}(\{1,a,b,h\})$ with an antisymmetric operation $[,] : \mathbf{o}\times\mathbf{o} \to \mathbf{o}$, satisfying the Jacobi-Identity and given by the terms $[b,a] := 1$, $[h,a] := a$ and $[h,b] := -b$. A change (*) of the algebraic basis of this vector space gives another form of the Lie algebra: Namely, $\mathbf{o} = \mathrm{span}_{\mathbb{C}}(\{1,x,p,h\})$ with $[x,p] = i1$, $[h,p] = ix$ and $[h,x] = -ip$. With these algebraic bases two real forms[24] of the complex Lie algebra can be found:

58 **STATEMENT.** ***(Oscillator Lie Algebra: Real Forms)*** *Within the oscillator Lie algebra $\mathbf{o}$ consider two sets $\{1,a,b,h\}$, and $\{i1,ix,ip,ih\}$. The two vector subspaces which are given by real linear spans of the sets and the appropriately restricted algebra multiplication are Lie algebras over the real numbers. These two real Lie algebras are non-isomorphic.*

Proof Indication: On the complex vector space $(\mathbf{o},+,\mathbb{C})$ define two anti-linear maps

[24] Antilinear maps are a generalization of the complex conjugation $\mathbb{C} \to \mathbb{C}$, with $\overline{\alpha+\beta\cdot\gamma} = \overline{\alpha}+\overline{\beta}\cdot\overline{\gamma}$ ($\alpha,\beta,\gamma \in \mathbb{C}$), to complex vector-spaces $(E,+,\mathbb{C})$: $J : E \to E$ with $J(x+s\cdot y) = J(x)+\overline{s}\cdot J(y)$ ($x,y \in E$ and $s \in \mathbb{C}$). The set of all fixed points $J(x) = x$ of such map is a vector-subspace over the real numbers. This kind of vector-subspace in an algebra is called *real form* if the algebra multiplication, restricted onto that subspace, remains defined.

by $J_1(\alpha 1+\beta a+\gamma b+\delta h) = \overline{\alpha}1+\overline{\beta}a+\overline{\gamma}b+\overline{\delta}h$ and $J_2(\alpha 1+\beta x+\gamma p+\delta h) = -\overline{\alpha}1-\overline{\beta}x-\overline{\gamma}p-\overline{\delta}h$, where the over-lines mean complex conjugation ($\alpha,\beta,\gamma,\delta \in \mathbb{C}$). Since compositions of these maps commute with the algebra multiplication $[,]\circ(J_i\times J_i) = J_i\circ[,]$ ($i=1,2$), the two sets of J_i-fix points $\{x \mid x\in \mathbf{o} \text{ and } J_i(x)=x\}$ are real Lie algebras. — The non-isomorphy of these real vector spaces is verified by using the multiplication tables: In the power sets of the two algebraic bases the following elements generate ideals $\{\emptyset\}$,$\{1\}$,$\{1,a\}$,$\{1,b\}$,$\{1,a,b\}$,$\{1,a,b,h\}$ and for the second algebraic basis $\{\emptyset\}$,$\{i1\}$,$\{i1,ix,ip\}$,$\{i1,ix,ip,ih\}$ those sets span ideals. So there are different numbers of ideals in both real subspaces. ∎

The real form $\mathrm{span}_{\mathbb{R}}(\{i1,ix,ip,ih\})$ can be given the coordinate notation $\left(\mathbb{R}^4,+,\mathbb{R}\right)$ with the algebra multiplication given by $[(\rho,\gamma,\chi,\tau),(\rho',\gamma',\chi',\tau')] := ((\chi\gamma'-\gamma\chi'),(\chi\tau'-\tau\chi'),(\tau\gamma'-\gamma\tau'),0)$ or, alternatively, consider the coordinate notation $(\mathbb{R}\times\mathbb{C}\times\mathbb{R})$ where the multiplication is given by $[(\rho,c,\tau),(\rho',c',\tau')] := (\mathrm{Im}(\overline{c}c'),i(c\tau'-\tau c'),0)$ (with $\mathrm{Im}(z) = \frac{1}{2i}(z-\overline{z})$).

59 **Note.** ***(Oscillator Lie Algebra in the Associative Heisenberg Algebra)*** *Now consider the vector subspace inside the associative Heisenberg algebra $C_1(\mathbf{h})$ (page 56), given by the complex linear span $\mathbf{o}' = \mathrm{span}_{\mathbb{C}}\left(\left\{1,a,b,a\,b+\frac{1}{2}\right\}\right)$. This vector subspace together with the commutator operation $[,]: \mathbf{o}'\times\mathbf{o}'\to\mathbf{o}'$, $(x,y)\mapsto x\,y-y\,x$ given by the associative multiplication is isomorphic to the oscillator Lie algebra $\mathbf{o}$.*

Now consider the set $\{a,b,h\}$ with the total order $a\le b\le h$. The centered enveloping algebra $C_1(\mathbf{o})$ of the oscillator Lie algebra $\mathbf{o} := \mathrm{span}_{\mathbb{C}}(\{1,a,b,h\})$ (The element 1 of the Lie algebra is associated with the neutral element of the free monoid.) is also called *associative oscillator algebra*. This algebra is given by the linear span $\mathrm{span}_{\mathbb{C}}(\mathrm{PBW}(\{a,b,h\}))$ which is generated by the set $\mathrm{PBW}(\{a,b,h\}) = \left\{a^k\,b^l\,h^m \,\middle|\, k,l,m\in\mathbb{N}_0\right\}$ of normally ordered elements and where the associative multiplication satisfies the commutation relations $b\,a-a\,b=1$, $h\,a-a\,h=a$ and $h\,b-b\,h=-b$. Similarly to the iteration of the Heisenberg commutation relation (Statement 54), the remaining commutation relations of the associative oscillator algebra can be iterated:

60 **Note.** ***(Oscillator Type Commutation Relation Iterated)*** *In the associative oscillator algebra $C_1(\mathbf{o})$ the iteration of the commutation relations $[h,c] = s\,c$, where the scalar $s := \begin{cases} 1 & \text{if } c=a \\ -1 & \text{if } c=b \end{cases}$ can take two values. (The proof is valid for any value of the scalar.) ($m,n\in\mathbb{N}_0$ and $n\neq 0$ and $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$):*

$$h^m\,c^n = \sum_{r=0}^{m}(sn)^r\begin{pmatrix} m \\ r \end{pmatrix} c^n\,h^{m-r}$$

To explicitly write down the multiplication of the associative oscillator algebra, the following generalized summation index changes are necessary:

61 **Note.** ***(More Summation Index Changes)*** *Consider a complete topological group $(G,+,\mathcal{O})$ taken to be commutative. And families $S:\mathbb{Z}^n\to G$ ($\mathbb{Z}$ denotes the module of*

integer numbers and $n \in \mathbb{N}$.) for which one of the multiple sums below exists. Then the following recommutations are valid:

$$\text{(i)} \qquad \sum_{r=0}^{\infty} \sum_{u=0}^{\infty} S(r,u) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} S(j, m-j)$$

$$\text{(ii)} \qquad \sum_{u=0}^{\infty} \sum_{w=0}^{r} S(u,w) = \sum_{m=0}^{\infty} \sum_{s=0}^{\min(m,r)} S(m-s, s) \quad (r \in \mathbb{N}_0)$$

$$\text{(iii)} \qquad \sum_{q=0}^{\infty} \sum_{t=1}^{\infty} S(q,t) = \sum_{l=1}^{\infty} \sum_{i=0}^{l-1} S(i, l-i)$$

$$\text{(iv)} \qquad \sum_{q=0}^{\infty} \sum_{m=0}^{\min(q,s)} S(q,m) = \sum_{l=0}^{\infty} \sum_{t=0}^{s} S(l+t, t) \quad (s \in \mathbb{N}_0)$$

$$\text{(v)} \quad \sum_{r=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{r} \sum_{w=0}^{r-v} S(r,u,v,w) = \sum_{m=0}^{\infty} \sum_{x=0}^{\infty} \sum_{y=0}^{x} \sum_{z=0}^{\min(m,y)} S(x, m-z, x-y, y-z)$$

$$\text{(vi)} \quad \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \sum_{m=0}^{\min(q,s)} S(p,q,s,t,m) = \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{l-1} \left(\left(\sum_{i=1}^{k} S(k-i, j, i, l-j, 0) \right) + \right.$$
$$\left. + \left(\sum_{n=1}^{\infty} \sum_{i=0}^{k} S(k-i, j+n, i+n, l-j, n) \right) \right)$$

The previous observations allow to explicitly formulate the multiplication of the associative oscillator algebra:

62 **NOTE.** ***(Associative Oscillator Multiplication)*** *Consider two elements $A, B \in C_1(\mathbf{o})$, given by families $(A_{k,l,m})_{k,l,m \in \mathbb{N}_0}, (B_{k,l,m})_{k,l,m \in \mathbb{N}_0} \in \mathbb{C}^{(\mathbb{N}_0^3)}$ of scalars and the associated linear combinations $A = \sum_{k,l,m \in \mathbb{N}_0} A(k,l,m)\, a^k\, b^l\, h^m$, $B = \sum_{k,l,m \in \mathbb{N}_0} B(k,l,m)\, a^k\, b^l\, h^m$. Then the algebra multiplication takes the following form:*

$$(A,B) \mapsto A\,B = \sum_{k,m \in \mathbb{N}_0} \left(c_0(k,0,m) + c_2(k,0,m)\right)\, a^k\, h^m +$$
$$+ \sum_{k,l,m \in \mathbb{N}_0, l \neq 0} \left(c_0(k,l,m) + c_1(k,l,m) + c_2(k,l,m) + c_3(k,l,m)\right)\, a^k\, b^l\, h^m$$

Where the coefficients are given as follows:

$$c_0(k,l,m) := \sum_{j=0}^{m} A(k,l,j)\, B(0,0,m-j)$$

$$c_1(k,l,m) := \sum_{r=0}^{\infty} \sum_{i=0}^{l-1} \sum_{j=0}^{\min(m,r)} (i-l)^{r-j} \begin{pmatrix} r \\ j \end{pmatrix} A(k,i,r)\, B(0, l-i, m-j)$$

$$c_2(k,l,m) = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \sum_{i=0}^{\min(k,s)} \sum_{j=0}^{\min(m,r)} \frac{s^{r-j}\, s!}{i!} \begin{pmatrix} r \\ j \end{pmatrix} \begin{pmatrix} l-i+s \\ l \end{pmatrix} \cdot$$
$$\cdot A(k-i, l-i+s, r)\, B(s, 0, m-j)$$

$$c_3(k,l,m) = \sum_{x=0}^{\infty} \sum_{y=0}^{x} \sum_{z=0}^{\min(m,y)} \frac{x!}{(x-y)!\,(y-z)!\,z!} \cdot \left(\sum_{j=0}^{l-1} (j-l)^{y-z} \left(\left(\sum_{i=1}^{k} i^{x-y}\, A(k-i,j,x)\, B(i,l-j,m-z) \right) + \right.\right.$$

$$\left.\left. + \left(\sum_{n=1}^{\infty} \sum_{i=0}^{k} \frac{(i+n)^{x-y}\,(i+n)!\,(j+n)!}{i!\,j!\,n!} \cdot A(k-i,j+n,x)\, B(i+n,l-j,m-z) \right) \right) \right)$$

Large and Huge Monoidal Algebras

This last chapter formulates an extension process for algebras with an algebraic basis. Then the extension process is realized three times: First, when introducing the so-called *large diagonal algebra* in which the eigenvalue problem of the harmonic oscillator can be solved. Then, there are constructed so-called *contour algebras* for calculating the Lie groups of the Heisenberg- and the oscillator algebras.

Extension

Consider a commutative field $\mathbb{K}$, a set B with countably infinitely many elements and a family of scalars $\left(t_{b,c,d}\right)_{b,c,d\in B} \in \mathbb{K}^{B\times B\times B}$, where the first index is finitely supported [AIII.10 §1.7 (7)]. Then the set $A := \operatorname{span}_{\mathbb{K}}(B) \cong \mathbb{K}^{(B)}$ is an algebra with respect to the vector addition and scalar multiplication defined component-wise and the algebra multiplication given by $\cdot : A\times A \to A,\ (x,y) \mapsto \sum_{b,c,d\in B} t_{b,c,d}\, x_c\, y_d\, b$ ($x = \sum_{c\in B} x_c\, c$ with $\left(x_c\right)_{c\in B} \in \mathbb{K}^{(B)}$ and $y = \sum_{d\in B} y_d\, d$ with $\left(y_d\right)_{d\in B} \in \mathbb{K}^{(B)}$). If the multiplication is associative, then for all $i,j,k,b \in B$ the relation $\sum_{c\in B} t_{b,i,c}\, t_{c,j,k} = \sum_{c\in B} t_{b,c,k}\, t_{c,i,j}$ (ASSOCIATIVITY) is valid.

Now, consider the algebra as a subspace of the product vector space $\mathbb{K}^B$. And consider the commutative field as a Hausdorff topological field, and the corresponding product topology $\mathcal{O}_\pi$ on the product space. If the topology of the field is first countable[25] and the index set B is countable, so is the product topology. (Recall that this product space is a vector space with a reduced basis B. See Example 33 (product) on page 33.)

In this space the algebra multiplication generally degenerates (Example 57 on page 57) to a simple relation

$$m = \left\{(x,y,z)\,|x,y \in \operatorname{sum}_{\mathcal{O}_\pi}(B) \text{ and } \exists z\left(z \in \operatorname{sum}_{\mathcal{O}_\pi}(B) \text{ and } \forall b\left(b\in B \Rightarrow z_b = \operatorname{pr}_b\left(\lim_{F\in\mathcal{F}_B} x_F\cdot y_F\right)\right)\right)\right\}$$

(where $\operatorname{pr}_b : \operatorname{sum}_{\mathcal{O}_\pi}(B) \to \mathbb{K}, x \mapsto x_b$, $x_F := \sum_{b\in F} x_b\, b$, $y_F := \sum_{b\in F} y_b\, b$ with $F \in \mathcal{F}_B$ where $\mathcal{F}_B$ is the set of all finite subsets of B). Nevertheless there are special cases, which can be treated with the statement that follows.

But first note the reason for differences in notation: If the algebraic basis B is countable and the topology of the commutative field is first countable (meaning the product topology $\mathcal{O}_\pi$ is first countable), then an element also can be written as a series $x = \lim_{n\in\mathbb{N}} \sum_{k=1}^{n} x_{b_k}\, b_k$ (namely the limit of partial sums) instead of the form of a linear sum $x = \lim_{F\in\mathcal{F}_B} \sum_{b\in F} x_b\, b$ (where $\mathcal{F}_B$ is the set of all subsets $F \subset B$ with finitely many elements). The last sentence is immediate by [WGT.71 10.4].

[25] A topology is called first countable if each neighborhood filter has a countable filter base [WGT.71 Chap. 4 10.3]

63 **Statement.** ***(Extension of Algebras)*** *Consider an (associative unital) algebra* $(A, +, \cdot, \mathbb{K})$ *with an algebraic basis* B*, as it has been introduced above. Consider this algebra* A *as a vector subspace inside the Hausdorff topological vector space* $\mathrm{sum}_{\mathcal{O}_\pi}(B)$*, which is given by an isomorphism* $\mathrm{sum}_{\mathcal{O}_\pi}(B) \cong \mathbb{K}^B$ *of vector spaces. Let there be a vector subspace* E *so that* $\mathrm{A} = \mathrm{span}(B) \subset E \subset \mathrm{sum}_{\mathcal{O}_\pi}(B)$ *which is supposed to make the (co-) restricted multiplication* $* := m\Big|_{E\times E}^{E}$ *relation defined. — Then follows that the vector subspace* E *is an (associative unital) algebra. The original algebra* $A \subset E$ *is a subalgebra of this algebra.*

Proof Indication: Consider an element $x \in E$, then there exists a family $(x_b)_{b\in B} \in \mathbb{K}^B$of scalars, so that the element is B-summable by this family. Consider analogous notations for other elements $y, z \in E$. As presupposed the relation m gives an operation $* : E \times E \to E$, therefore these elements can be multiplied to $(x+y)*z, x*z, y*z \in E$. By the continuity of the map pr_b ($b \in B$) with respect to the topology of point-wise convergence follows $(x + y) * z = \lim_{F\in\mathcal{F}_B}(x_F + y_F) \cdot z_F$. From the distributivity of the algebra A and the continuity of the vector addition in $\mathrm{sum}_{\mathcal{O}_\pi}(B)$ follows the equality $\lim_{F\in\mathcal{F}_B}(x_F \cdot z_F) + \lim_{F\in\mathcal{F}_B}(y_F \cdot z_F) = x*z+y*z$. So the operation $* : E \times E \to E$ satisfies the distributivity.

As presupposed, the elements above can be multiplied successively with $(x * y) * z, x * (y * z) \in E$. So the corresponding families of scalars are B-summable and by using the associativity (ASSOCIATIVITY) of the algebra multiplication in A the operation $* : E \times E \to E$ can be recognized as associative. ■

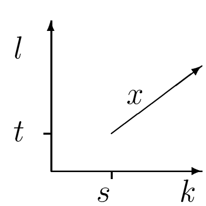
Large Diagonal Algebra

Consider the associative Heisenberg algebra $C_1(\mathbf{h}) = \mathrm{span}_{\mathbb{C}}(\mathrm{PBW}(\{a, b\}))$, as introduced on page 56. For this algebra consider the (complete) first countable Hausdorff topological vector space $\mathrm{sum}_{\mathbb{C},\mathcal{O}_\pi}(\mathrm{PBW}(\{a, b\})) \cong \mathbb{C}^{\mathrm{PBW}(\{a,b\})}$ can be seen as a super-space.

64 **Definition** ***(Diagonal Elements)*** *Inside the previously mentioned linear sum define*

$$D_{a,b} := \left\{ \sum_{n\in\mathbb{N}_0} x_n\, a^{n+s}\, b^{n+t} \,\middle|\, (x_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \text{ and } s, t \in \mathbb{N}_0 \right\} \subset \mathrm{sum}_{\mathbb{C},\mathcal{O}_\pi}(\mathrm{PBW}(\{a, b\}))$$

to be the set of diagonal elements.



With this set, a vector super-space of the associative Heisenberg algebra can be constructed, which extends the associative multiplication:

65 **STATEMENT.** ***(Multiplication of the Large Diagonal Algebra)*** *The multiplication operation of the associative Heisenberg algebra* $C_1(\mathbf{h}) = \text{span}_{\mathbb{C}}\left(\text{PBW}\left(\{a,b\}\right)\right)$ *gives a relation* m *on the linear sum* $\text{sum}_{\mathbb{C},\mathcal{O}_\pi}\left(\text{PBW}\left(\{a,b\}\right)\right)$*. This multiplication relation is defined on the linear span of the diagonal elements* $\text{span}_{\mathbb{C}}\left(D_{a,b}\right)$*.*

Proof Indication: For an element $x = \sum_{k,l\in\mathbb{N}_0} x_{k,l}\, a^k\, b^l$ in the linear span of the diagonal elements $\text{span}\left(D_{a,b}\right)$ the families $\left(x_{s,n}\right)_{n\in\mathbb{N}_0}, \left(x_{n,s}\right)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ ($s \in \mathbb{N}_0$) have only finitely many non-zero coordinates, which means those families are finitely supported. Consider another element y in the same linear span with coordinates $\left(y_{k,l}\right)_{k,l\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$. Then the family $\left(x_{k-q,n+p}\, y_{n+q,l-p}\right)_{q\in\{0,\ldots,k\};p\in\{0,\ldots,l\};n\in\mathbb{N}_0}$ ($k,l \in \mathbb{N}_0$) that describes the multiplication is also finitely supported in the same way. ■

By the Statement 63 the linear span $\text{span}_{\mathbb{C}}\left(D_{a,b}\right)$ is an associative unital algebra, the so-called *large diagonal algebra* $DC_1(\mathbf{h})$, of which the associative Heisenberg algebra $C_1(\mathbf{h})$ is a subspace. The set $\text{PBW}\left(\{a,b\}\right)$ is *not* a reduced basis in the large diagonal algebra. Nevertheless elements of this algebra $DC_1(\mathbf{h}) = \text{span}_{\mathbb{C}}\left(D_{a,b}\right)$ are written as linear sums in $\text{sum}_{\mathbb{C},\mathcal{O}_\pi}\left(\text{PBW}\left(\{a,b\}\right)\right)$.

In the large diagonal algebra the quantum-theoretic eigenvalue problem

$$h\, x_\lambda = \lambda\, x_\lambda$$

for the energy function $h = a\,b + \frac{1}{2}$ (also called Hamilton function or Hamiltonian) of the harmonic oscillator can be solved[26]. Here the scalar eigenvalues λ and the respective eigenelements x_λ are unknown. Rewriting the eigenproblem as $(h - \lambda 1)\, x_\lambda = 0$ shows that those scalars λ for which the algebra element $(h - \lambda 1)$ is invertible cannot be potential eigenvalues. So eigenvalues always are elements of the spectrum:

66 **DEFINITION** ***(Spectrum)*** *In an associative unital algebra over a field* $\mathbb{K}$ *consider an element* h *and the neutral element* 1 *of the multiplication. The set of scalars*

$$\text{spec}\,(h) := \{s\,|\,s \in \mathbb{K}\ \text{and}\ (h - s1)\ \text{is NOT multiplicatively invertible}\}$$

is called spectrum of the element h *or if no confusion arises simply* spectrum*.*

The results of a concrete calculation are compiled in the following statement:

67 **STATEMENT.** ***(Oscillator Eigenvalue Problem)*** *In the large diagonal algebra* $DC_1(\mathbf{h})$ *over the complex numbers the following is true: (Spectrum) The spectrum of the energy function* h *is given by the set of scalars* $\text{spec}\,(h) = \left\{ n + \frac{1}{2} \middle| n \in \mathbb{N}_0 \right\}$*. (Eigenelements) Each spectral element has a related eigenelement* $x_n = c_n \sum_{k\in\mathbb{N}_0} \frac{(-1)^k}{k!}\, a^{k+n}\, b^k$ *with* $c_n \in \mathbb{C}\backslash\{0\}$*, so that the relation* $h\, x_n = (n + \frac{1}{2})\, x_n$ *is satisfied for natural numbers* $n \in \mathbb{N}$*. In the case of* $n = 0$ *the eigenelement* $x_0 = \sum_{l\in\mathbb{N}_0;k\leq l} c_{0,l-k} \frac{(-1)^k}{k!}\, a^k\, b^l$ *(*$h\, x_0 = \frac{1}{2} x_0$*) has more freedom* $\left(c_{0,m}\right)_{m\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}\backslash\{0\}$ *than a single scalar factor. (Creation- and Annihilation Relations) In those cases where the coefficients* c_n *and* $c_{0,0}$ *are equal and* $c_{0,n} = 0$ *for* $n \in \mathbb{N}$*, the creation and annihilation relations are valid:* $a\, x_n = x_{n+1}$ *and* $b\, x_{n+1} = (n+1)\, x_n$ *(*$n \in \mathbb{N}_0$*)*

[26] In this formulation of the eigenvalue problem the Hamilton function is not an operator, but just another element of the algebra.

Proof Indication: (Spectrum) Show that inside the algebra $DC_1(\mathbf{h})$ the elements $h - \lambda 1 = a\,b - \kappa 1$, $\sum_{n\in\mathbb{N}_0} \frac{-1}{\kappa(\kappa-1)\cdot\ldots\cdot(\kappa-n)}\, a^n\, b^n$ $(\kappa = \lambda - 1/2)$ are a pair of multiplicative inverses: Therefore consider the product of a linear sum $\sum_{k',l'\in\mathbb{N}_0} c_{k',l'}\, a^{k'}\, b^{l'}$ and the element $a\,b - \kappa 1$: (1) $\sum_{k',l'\in\mathbb{N}_0} c_{k',l'}\,(aba^{k'}b^{l'} - \kappa\, a^{k'}\, b^{l'})$ With the Heisenberg commutation relation in the form $\left[ab, a^{k'}\right] = k'\, b^{k'}$(where the bracket $[x,y]$ stands for the commutator $xy - yx$) and the reassociation of sums with infinitely many summands, which is only possible in complete groups [TGIII.39 §5.3 Th. 2] (here $\mathrm{sum}_{\mathbb{C},\mathcal{O}_\pi}\,(\mathrm{PBW}\,(\{a,b\}))$), the Term (1) changes to linear sums that contain only normally ordered monoid elements: (2) $\sum_{k',l'\in\mathbb{N}_0} c_{k',l'}\, a^{k'+1} b^{l'+1} - \sum_{k',l'\in\mathbb{N}_0} c_{k',l'}\,(\kappa - k')\, a^{k'}\, b^{l'}$ Gather contributions to the same monoid elements by changing the summation indices ($k = k' + 1$ and $l = l' + 1$ in the first sum, $k = k'$ and $l = l'$ in the second sum of Term (2)): (3) $\sum_{k,l\geq 1} c_{k-1,l-1}\, a^k b^l - \sum_{k,l\in\mathbb{N}_0} c_{k,l}\,(\kappa - k)\, a^k b^l$ Now decompose the second sum of Term (3) into the following four summands, $\sum_{k\geq 1,l\geq 1}\ldots + \sum_{k\geq 1,l=0}\ldots + \sum_{k=0,l\geq 1}\ldots + \sum_{k=l=0}\ldots$ and rewrite the Term (3) in its unique representation as a sum of infinitely many normally ordered monoid elements: $\sum_{k,l\geq 1}(c_{k-1,l-1} - c_{k,l}(\kappa-k))\, a^k b^l - \sum_{k\geq 1,l=0} c_{k,0}(\kappa - k)\, a^k - \sum_{k=0,l\geq 1} \kappa\, c_{0,l}\, b^l - \kappa\, c_{0,0} 1$ Then by setting this result equal to the neutral element 1 of the algebra multiplication, comparing coefficients yields $c_{0,0} = -1/\kappa$, $c_{0,l} = 0$ for $l > 0$, $c_{k,0} = 0$ for $k > 0$ and by iteration $c_{k,l} = \frac{c_{k-m,l-m}}{\prod_{m'\in\{k-m+1,\ldots,k\}}(\kappa - m')}$ $(m \in \mathbb{N})$. So all complex numbers $\lambda \in \mathbb{N}_0 + \frac{1}{2}$ constitute the spectrum of the Hamilton function h. With respect to the calculations already shown, the remaining cases (Eigenelements) and (Creation- and Annihilation Relations) are straightforward calculations. ■

The notion of a *-algebra[27] together with the so-called trace relation are other means (other than eigenvalue problems) for extracting numbers out of algebraic relations of monoid elements.

68 **NOTE.** ***(*-Algebra)*** *Consider the large diagonal algebra $DC_1(\mathbf{h})$ and the map*

$$+ : DC_1(\mathbf{h}) \to DC_1(\mathbf{h}),$$

$\sum_{k,l\in\mathbb{N}_0} x_{k,l}\, a^k b^l \mapsto \sum_{k,l\in\mathbb{N}_0} \overline{x_{k,l}}\, a^l b^k$ *which is also given by the setting $b^+ := a$ and $a^+ := b$. This map makes the large diagonal algebra a *-algebra.*

The *trace-relation* of an element can be defined as the sum of all diagonal coefficients:

$$\mathrm{Tr}\left(\sum_{k,l\in\mathbb{N}_0} A_{k,l}\, a^k b^l\right) = \sum_{n\in\mathbb{N}_0} A_{n,n}$$

Consider the solution of the eigenvalue problem of the harmonic oscillator. Consider its energy eigenelements x_n for a fixed value $c \in \mathbb{C}$ with $\forall n\,(n \in \mathbb{N}_0 \Rightarrow c_n = c)$ and $\forall m\,(m \in \mathbb{N} \Rightarrow c_{0,m} = 0)$. Then a calculation which makes use of the creation- and annihilation relations yields: $\mathrm{Tr}\,(x_n\, x_m^+) = \frac{|c|^2}{e}\,\delta_{n,m}$, where the $\delta_{n,m}$ represents the Kronecker-delta $\delta_{n,m} := \begin{cases} 1 & \text{if}\, n = m \\ 0 & \text{if}\, n \neq m \end{cases}$ and the letter "e" represents the value $e = \exp(1)$ of the

[27] A *-algebra $(A, +, \cdot, \mathbb{C})$ is characterized by a unary map $+ : A \to A$ that is an idempotent $\left(x^+\right)^+ = x$, antilinear $(x + y)^+ = x^+ + y^+$, $(sx)^+ = \overline{s}\,x^+$and treats algebra products like the multiplicative inversion $(xy)^+ = y^+x^+(x, y \in A,\, s \in \mathbb{C}$ and the overline denotes the conjugate of a complex number)

exponential function. From this result follow the relations $\mathrm{Tr}\,(a^p\,x_n\,x_m^+) = \frac{|c|^2}{e}\,\delta_{n+p,m}$ and $\mathrm{Tr}\,(b^p\,x_n\,x_m^+) = \frac{n!}{(n-p)!}\frac{|c|^2}{e}\,\delta_{n-p,m}$. For the eigenelements x_n the *expectation-relation* of an element $z \in DC_1(\mathbf{h})$ is given by

$$\langle z \rangle_n := \mathrm{Tr}\,(z\,x_n\,x_n^+)\,.$$

This establishes the well-known [SAKU.94 (2.3.40)] expectation values and the uncertainty relation for the quantum theoretical harmonic oscillator:

69 **NOTE.** ***(Expectation Values of the Oscillator)*** *Consider the large diagonal *-algebra $DC_1(\mathbf{h})$, together with the trace-relation and the expectation-relation. Using the creator- and annihilator relations follows:*

$$\langle x \rangle_n = \langle p \rangle_n = 0$$
$$\langle x^2 \rangle_n = \langle p^2 \rangle_n = \sqrt{\langle x^2 \rangle_n}\sqrt{\langle p^2 \rangle_n} = \left(n + \tfrac{1}{2}\right)\,\frac{|c|^2}{e}$$

Huge Contour Algebra

Recall the associative Heisenberg algebra $C_1(\mathbf{h})$ and the associative oscillator algebra $C_1(\mathbf{o})$. Each of these can be extended to a so-called huge algebra.

The vector space structure, given by the associative Heisenberg algebra $C_1(\mathbf{h})$, has the product topological completion $\mathrm{V}_{\mathbf{h}} := \mathrm{sum}_{\mathbb{C},\mathcal{O}_\pi}\,(\mathrm{PBW}\,(\{a,b\}))$. Therein consider a subset which is given by a contour restriction as follows:

$$E_{\mathbf{h}} = \left\{ X \,\middle|\, X = \sum_{k,l\in\mathbb{N}_0} X_{k,l}\,a^k b^l \in V_{\mathbf{h}} \,\mathrm{and}\, \exists c_x, d_x \in \mathbb{R}^{\geq 0} \,\mathrm{and}\, |X_{k,l}| \leq c_x\,\frac{d_x^{(k+l)}}{k!\,l!} \right\}$$

$(s \in \mathbb{R}^{\geq 0} \Leftrightarrow s \in \mathbb{R}\,\mathrm{and}\,\mathrm{s} \geq 0)$

70 **STATEMENT.** ***(Huge Heisenberg Algebra)*** *The set $E_{\mathbf{h}}$ can be given the structure of an associative unital algebra so that the associative Heisenberg algebra $C_1(\mathbf{h})$ is a subspace.*

Proof Indication: The binomial expansion of real numbers and the triangle equation of the absolute value makes the vector addition and scalar multiplication inside E_h defined. The values $c_{x\cdot y}$, $d_{x\cdot y} \in \mathbb{R}^{\geq 0}$ which dominate the absolute values of the multiplication relation are given by $c_{x\cdot y} := c^2\,e^{d^2}$ and $d_{x\cdot y} = 2\,d$ with $c := \max\,(c_x, c_y)$ and $d := \max\,(d_x, d_y)$. The rest follows from the Statement 63 on page 62 about the extension of algebras. ■

This algebra is called *huge contour algebra* of the (associative) Heisenberg algebra. This is not a large algebra because in the term of the multiplicative operation individual coefficients may be sums of more than finitely many scalars. This new aspect is cast into a definition:

71 **Definition** ***(Huge Algebra)*** *Consider an algebra $(A, +, \cdot, \mathbb{K})$ which is equal to or a subspace of a Hausdorff topological vector space $(F, +, \mathbb{K})$ with a reduced basis B (mostly this super-space is a linear sum $F = \mathrm{sum}_{\mathbb{K},\mathcal{O}_\pi}(B)$ with respect to the product topology induced from the isomorphic space $\left(\mathbb{K}^B, +, \mathbb{K}, \mathcal{O}_\pi\right)$). Let the algebra multiplication of two elements $x, y \in A$ be writable as a linear sum $x \cdot y = \sum_{b \in B} f_b(x, y)\, b \in A$. If the coefficients $f_b(x, y)$ are defined by a limit process in the topological field of the algebra, then the algebra is called a huge algebra.*

("Small" algebras have the set B as an algebraic basis, the family $(f_b(x, y))_{b\in B} \in \mathbb{K}^B$ is finitely supported and the coefficients $f_b(x, y)$ contain no limit process. In "large" algebras the coefficients $f_b(x, y)$ contain no limit process, but the family may contain infinitely many non-zero entries; [AIII.27 §2.10] defines the term "large algebra".) The Theorem of Wielandt [RUD.332 13.6] states that the large diagonal algebra $DC_1(\mathbf{h})$ and the huge contour algebra of the Heisenberg algebra cannot be normed so that the multiplication becomes continuous.

In the huge Heisenberg algebra it is possible to exponentiate the generators of the underlying free monoid and to multiply the exponentials: $e^{s\,a}\, e^{t\,b}\, e^{s'\,a}\, e^{t'\,b} = e^{s'\,t\,1} e^{(s+s')\,a} e^{(t+t')\,b}$ $(s, t, s', t' \in \mathbb{C})$ This gives an operation on the product set $\mathbb{C}^3$:

72 **Note.** ***(Heisenberg Lie Group)*** *The operation $(r, s, t)(r', s', t') = (r+r'+s't, s+s', t+t')$ makes the product set $\mathbb{C}^3$ a parametrization of the Lie group of the Heisenberg Lie algebra $\mathbf{h}$.*

Now consider the case of the associative oscillator algebra $C_1(\mathbf{o})$, its vector space structure has the product topological completion $\mathrm{V}_{\mathbf{o}} := \mathrm{sum}_{\mathbb{C},\mathcal{O}_\pi}(\mathrm{PBW}(\{a, b, h\}))$. Therein consider a subset, given by a contour restriction as follows:

$$E_{\mathbf{o}} = \left\{ X \,\middle|\, X = \sum_{k,l,m\in\mathbb{N}_0} X_{k,l,m}\, a^k b^l h^m \in V_{\mathbf{o}} \text{ and } \exists c_x, d_x \in \mathbb{R}^{\geq 0} \text{ and } |X_{k,l,m}| \leq c_x\, \frac{d_x^{(k+l+m)}}{k!\, l!\, m!} \right\}$$

$(s \in \mathbb{R}^{\geq 0} \Leftrightarrow s \in \mathbb{R} \text{ and } s \geq 0)$

73 **Statement.** ***(Huge Oscillator Algebra)*** *The set $E_{\mathbf{o}}$ can take the structure of an associative unital algebra so that the associative oscillator algebra $C_1(\mathbf{o})$ is a subspace.*

This algebra is called *huge contour algebra* of the (associative) oscillator algebra.

Proof Indication: The proof is similar to the Heisenberg case, therefore only differing aspects are mentioned: The values $c_{x\cdot y}$, $d_{x\cdot y} \in \mathbb{R}^{\geq 0}$ which dominate the absolute values of the multiplication relation are given by $c_{x\cdot y} := 8c^2\, e^{d^2}(1 + \exp(d^2 \exp(d)))$ and $d_{x\cdot y} = 8d\, e^{2(d+2)}$ with $c := \max(c_x, c_y)$ and $d := \max(d_x, d_y)$. In this process are used inequality relations like the following one $\sum_{i=0}^{k} e^{id} \leq (k+1)e^{kd}$ ($d \geq 0$ and $k \in \mathbb{N}_0$). ■

This algebra is called *huge contour algebra* of the (associative) oscillator algebra. In this algebra the relation $e^U\, e^V\, e^{-U} = \exp\left(\sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}(U))^k\right)(V)$ can be directly verified ($\mathrm{ad}(U)$ is given by $X \mapsto UX - XU$ (set $V = a$ or $V = b$ and set $U = h$ otherwise this formula is a special case [LIEII.90 §6 Exercise 1] of the Baker-Campbell-Hausdorff formula)). This gives the following relation:

$e^{p1} e^{xa} e^{yb} e^{th}\, e^{p'1} e^{x'a} e^{y'b} e^{t'h} = e^{(p+p'+yx'e^t)1}\, e^{(x+x'e^t)}\, e^{(y+y'e^{-t})b}\, e^{(t+t')h}$ $(p, x, y, t, p', x', y', t' \in \mathbb{C})$

This yields the note:

74 **NOTE.** ***(Oscillator Lie Group)*** *The operation* $(p,x,y,t)(p',x',y',t') = (p+p'+yx'e^t, x+x'e^t, y+y'e^{-t}, t+t')$ *makes the product set* $\mathbb{C}^4$ *a parametrization of the Lie group of the oscillator Lie algebra* $\mathbf{o}$.

Text (ISBN 3-00-004696-8) to be found at http://www.schmarsow.de

Back Matter

Supplement - Elementary Lie Algebra

A simple case of a Lie algebra is given by the following associative algebra: Consider a set of two elements with a total order $(\{a, b\}, \leq)$ and a commutative field $\mathbb{K}$. To be able to work with the commutation relation $b \cdot a - a \cdot b = s\, a$ ($s \in \mathbb{K}$) consistently, consider the Lie algebra $\mathbf{a}_s = (\mathrm{span}_{\mathbb{K}}(\{a, b\}), [b, a] = s\, a)$ and the associated universal enveloping algebra $\mathrm{U}(\mathbf{a}_s)$. The Poincaré-Birkhoff-Witt Theorem, then states that the linear span of the Poincaré-Birkhoff-Witt set $\mathrm{span}_{\mathbb{K}}(\mathrm{PBW}(\{a, b\}))$ together with the commutation relation above has an algebra structure which makes it isomorphic to the universal enveloping algebra.

The iterated commutation relation has been presented in Statement 60 on page 58, so with this information the multiplication of the associative algebra can be made explicit: Two elements $A = \sum_{p,q \in \mathbb{N}_0} A_{p,q}\, a^p b^q$ and $B = \sum_{r,s \in \mathbb{N}_0} B_{r,s}\, a^r b^s$ ($A_{p,q}, B_{r,s} \in \mathbb{K}$) multiplied together give the product:

$$A \cdot B = \sum_{p,q,r,s \in \mathbb{N}_0} A_{p,q}\, B_{r,s}\, a^p b^q a^r b^s$$

To apply the iterated commutation relation ($ba - ab = s\, a \Rightarrow b^q a^r = \sum_{t=0}^{q} (sr)^t \begin{pmatrix} q \\ t \end{pmatrix} a^r b^{q-t}$, $q, r \in \mathbb{N}_0, r \neq 0$) recommute the summands of the product term:

$$A \cdot B = \sum_{p,q,s \in \mathbb{N}_0} A_{p,q}\, B_{0,s}\, a^p b^q b^s + \sum_{p,q,r,s \in \mathbb{N}_0, r \neq 0} A_{p,q}\, B_{r,s}\, a^p b^q a^r b^s$$

Then the reordered result is:

$$A \cdot B = \sum_{p,q,s \in \mathbb{N}_0} A_{p,q}\, B_{0,s}\, a^p b^{q+s} + \sum_{p,q,r,s \in \mathbb{N}_0, r \neq 0} \sum_{t=0}^{q} (sr)^t \begin{pmatrix} q \\ t \end{pmatrix} A_{p,q}\, B_{r,s}\, a^{p+r} b^{q-t+s}$$

Prepare to collect the coefficients of equal monoid elements by applying two generalized index transformations
($\sum_{q,s \in \mathbb{N}_0} S(q, s) = \sum_{l \in \mathbb{N}_0} \sum_{i=0}^{l} S(i, l - i)$ and
$\sum_{p,q,r,s \in \mathbb{N}_0, r \neq 0} \sum_{t=0}^{q} S(p, q, r, s, t) = \sum_{k,l,m \in \mathbb{N}_0, k \neq 0} \sum_{i=1}^{k} \sum_{j=0}^{\min(l,m)} S(k - i, m, i, l - j, m - j)$)
so that the monoid elements have single indices as their exponents:

$$A \cdot B = \sum_{k,l \in \mathbb{N}_0} \sum_{i=0}^{l} A_{k,i}\, B_{0,l-i}\, a^k b^l + \sum_{k,l,m \in \mathbb{N}_0, k \neq 0} \sum_{i=1}^{k} \sum_{j=0}^{\min(l,m)} (si)^{m-j} \begin{pmatrix} m \\ m - j \end{pmatrix} A_{k-i,m}\, B_{i,l-j}\, a^k b^l$$

Finally collect the coefficients:

$$A \cdot B =$$
$$= \sum_{l\in\mathbb{N}_0} \sum_{i=0}^{l} A_{0,i}\, B_{0,l-i}\, b^l +$$
$$+ \sum_{k,l\in\mathbb{N}_0,k\neq 0} \left(\left(\sum_{i=0}^{l} A_{k,i} B_{0,l-i} \right) + \sum_{m\in\mathbb{N}_0} \sum_{i=1}^{k} \sum_{j=0}^{\min(l,m)} (si)^{m-j} \begin{pmatrix} m \\ j \end{pmatrix} A_{k-i,m}\, B_{i,l-j} \right) a^k b^l$$

The huge contour algebra of this associative algebra has the same form like that of the Heisenberg algebra except that the coefficient takes another form
$d_{x\cdot y} = d\left(2 + e^{\max(\ln(2)+sd,1+\ln(2))} + e^{1+sd}\right)$ ($d := \max(d_y, d_y)$, $c := \max(c_x, c_y)$).

Supplement - Rotator Lie Algebra

In quantum-physics rotational symmetry produces observables, which in special cases are similar to the classical angular momentum of a rotating massive object [BÖHM.95 III.2 and III.3][FICK.289 Part 4 Chap. 3][SAKU.152 Chap. 3]. Those observables can be described mathematically by elements of the so-called *rotator Lie algebra*:

Consider a set $\{J_x, J_y, J_z\}$ of three elements and the field $\mathbb{C}$ of the complex numbers. The set $\text{span}_{\mathbb{C}}(\{J_x, J_y, J_z\})$ of all linear combinations of these elements is a vector space with respect to component-wise addition and scalar multiplication. On this vector space can be defined an algebra multiplication using the relations $[J_x, J_y] = iJ_z$, $[J_z, J_x] = iJ_y$ and $[J_y, J_z] = iJ_x$ (where two of them emerge from cyclic permutation of the remaining relation) and requiring the multiplication to be antisymmetric. Straight-forward calculations prove this space to be a Lie algebra:

$$(\text{span}_{\mathbb{C}}(\{J_x, J_y, J_z\}), +, [J_x, J_y] = iJ_z \text{ cyclic and antisymmetric})$$

A basis transformation ($J_+ := s(J_x + iJ_y)$, $J_- := s(J_x - iJ_y)$ with $s \in \mathbb{C}\backslash\{0\}$) gives another form of the rotator Lie algebra:

$$\left(\text{span}_{\mathbb{C}}(\{J_+, J_-, J_z\}), +, [J_+, J_-] = 2s^2 J_z\ \ [J_+, J_z] = -sJ_+, [J_-, J_z] = sJ_- \text{and antisymmetric}\right)$$

This Lie algebra is used to define its universal enveloping algebra in which commutators take the role of the bracket relations above. The Poincaré-Birkhoff-Witt Theorem shows that the rotator Lie algebra is a subalgebra given by commutators in its universal enveloping algebra and that this universal enveloping algebra is isomorphic to the linear span of the Poincaré-Birkhoff-Witt set $\text{PBW}(\{J_+, J_-, J_z\})$. To ease the notation the following change of symbols is used henceforth: $a := J_+$, $b := J_-$and $c := J_z$ and $s = 1$. In this notation the Euklidean scalar product of the angular momentum $\underline{J} = (J_x, J_y, J_z)$ with itself becomes: $\underline{J}^2 = J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{2}(ab + ba) + c^2$

Furthermore, elementary induction gives the following iterated commutation relations (here, as generally, the square brackets in an associative algebra are used as a shorthand for a commutator: $[x, y] := x \cdot y - y \cdot x$ (in associative algebras!)):

$$[a, c] = -a \Rightarrow \qquad [a^n, c] = -na^n$$

$$
\begin{aligned}
[b,c] = b \Rightarrow &\qquad [b^n, c] = +nb^n \\
[a,b] = \mu c \Rightarrow &\quad [a^n, b] = n\mu a^{n-1}\left(c + \tfrac{n-1}{2}1\right) \quad \text{and} \\
&\quad [a, b^n] = n\mu b^{n-1}\left(c - \tfrac{n-1}{2}1\right)
\end{aligned}
$$

(Where in showing the last two relations has been used the formula $\forall n \left(n \in \mathbb{N} \Rightarrow \sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}\right)$) A generalized version of the commutation relation $[a,b] = \mu c$ is given by

$$b^m a^n = \sum_{l=0}^{m} \left(\left(-\frac{\mu}{2}\right)^l \binom{n}{l} \binom{m}{l} l!\, a^{n-l} b^{m-l} \left(\sum_{k=0}^{l} \frac{2^k}{k!} \left(\frac{\partial}{\partial n}\right)^k \prod_{r=0}^{l-1} (n-m+r)\, c^k \right) \right)$$

which needn't be used in the following consideration.

In search of a solution to the eigenvalue problem $J^2 x = \lambda x$ the following equation should be solved: $(J^2 - \lambda 1) \cdot \sum_{k,l,m \in \mathbb{N}_0} c_{k,l,m}\, a^k b^l c^m = 1$ The first step in this process is to rewrite the product in normally ordered monoid elements. This is possible using the commutation relations above supplemented by $\left[c^2, a^k\right] = k^2\, a^k + 2k\, a^k c$ and by $\left[c^2, b^l\right] = l^2\, b^l - 2l\, b^l c$. Elementary changes of summation indices makes it possible to collect coefficients of the same monoid elements. — Then setting the product equal to one, yields a collection of interrelated recursion relations and starting points for these relations.

A kind of generalized Fibonacci sequence inside a commutative ring $(R, +, \cdot)$ is used multiply ($\alpha, \beta, F_0, F_1 \in R \backslash \{0\}$ and $\gamma_1 = 0$ and $\forall n\, (n \geq 2 \Rightarrow \gamma_n \in R)$):

$$F_{n+2} = \alpha\, F_{n+1} + \beta\, F_n + \gamma_{n+2}$$

giving

$$
\begin{aligned}
F_n = &\left(\textstyle\sum_{k=0}^{2k+1 \leq n} c_{n-1,k}\, \alpha^{n-2k-1} \beta^k\right) F_1 + \\
+ &\left(\textstyle\sum_{k=0}^{2k+2 \leq n} c_{n-2,k}\, \alpha^{n-2k-2} \beta^{k+1}\right) F_0 + \textstyle\sum_{m=1}^{n} \left(\sum_{k=0}^{2k+1 \leq m} c_{m-1,k}\, \alpha^{m-2k-1} \beta^k\right) \gamma_{n-m+1}
\end{aligned}
$$

with a family of coefficients $(c_{n,k})_{n,k \in \mathbb{N}_0}$ given by the relations $c_{n,0} = c_{2k,k} = 1$, $c_{2k-1,k} = 0$ and the recursion relation $c_{n,k} = c_{n-1,k} + c_{n-2,k-1}$.

This yields $(J^2 - \lambda 1) \cdot \sum_{k,l,m \in \mathbb{N}_0} c_{k,l,m}\, a^k b^l c^m =$

$$
\begin{aligned}
&\textstyle\sum_{k \in \mathbb{N}_0} (-1)^k \prod_{s=0}^{k} \frac{1}{\frac{\mu}{2} s(s+1) - \lambda} a^k b^k + \\
&\textstyle\sum_{k \in \mathbb{N}_0} (-1)^{k+1} \left(\prod_{s=0}^{k} \frac{1}{\frac{\mu}{2} s(s+1) - \lambda}\right) \left(\sum_{t=0}^{k} \frac{-\frac{\mu}{2}(2t+1)}{\frac{\mu}{2} t(t+1) - \lambda}\right) a^k b^k c + \\
&\textstyle\sum_{k \in \mathbb{N}_0, m \geq 2} (-1)^{k+1} \left(\prod_{s=0}^{k} \frac{1}{\frac{\mu}{2} s(s+1) - \lambda}\right) \left(\sum_{s=0}^{2s \leq m} c_{m,s} \frac{\left(-\frac{\mu}{2}\right)^{m-2s}}{\lambda^{m-s}} + \right. \\
&\textstyle\left(\sum_{t=0}^{k} \frac{-\frac{\mu}{2}(2t+1)}{\frac{\mu}{2} t(t+1) - \lambda}\right) \left(\sum_{s=0}^{2s+1 \leq m} c_{m-1,s} \frac{\left(-\frac{\mu}{2}(2k+1)\right)^{m-2s-1}}{\left(\lambda - \frac{\mu}{2} k(k+1)\right)^{m-s-1}}\right) + \\
&\textstyle\left. \frac{1}{\frac{\mu}{2} k(k+1) - \lambda} \left(\sum_{s=0}^{2s+2 \leq m} c_{m-2,s} \frac{\left(-\frac{\mu}{2}(2k+1)\right)^{m-2s-2}}{\left(\lambda - \frac{\mu}{2} k(k+1)\right)^{m-s-2}}\right)\right) a^k b^k c^m
\end{aligned}
$$

This clearly shows the spectrum of the square value of the angular momentum to be discrete, as follows:

$$\operatorname{spec}\left(J^2\right) = \left\{\lambda \,\middle|\, \lambda = \frac{\mu}{2}k(k+1) \text{ and } k \in \mathbb{N}_0\right\}$$

Since the factor $\mu = 2$ the result is what could be expected from spectrum calculations in associative algebras.

Bibliography

In the text, the reference symbols usually are supplemented by chapter and page number.

[BÖHM] Böhm, Arno. *Quantum Mechanics*. New York: Springer, 1979.

[A] Bourbaki, Nicolas. *Algèbre*. Chap. I-III. Paris: Hermann, 1970.

[AC] —. *Commutative Algebra* [*Algèbre Commutative*]. Chap. III, Paris: Hermann, 1972.

[EVT] —. *Espaces Vectoriels Topologiques*. Deuxième edition. Chap. I,II. Paris: Hermann, 1965.

[EVT] —. *Espaces Vectoriels Topologiques*. Chap. III,VI,V. Paris: Hermann, 1967.

[LIE] —. *Groupes et Algèbres de Lie*. Chap. I (1971), Chap. II (1972), Paris: Hermann.

[E] —. *Théorie des Ensembles*. Paris: Hermann, 1970.

[TG] —. *Topologie Générale*. Chap. I-IV. Paris: Hermann, 1971.

[TG] —. *General Topology* [*Topologie Générale*]. Chap. V-X. Paris: Hermann, 1966.

[DIX] Dixmier, Jacques. *Algèbres Enveloppantes*. Paris: Gauthier-Villars, 1974.

[ET] Erné, Marcel. “Topologie”. Vorlesungsausarbeitung. Universität Hannover, 1984.

[FICK] Fick, Eugen. *Einführung in die Grundlagen der Quantentheorie*. Fifth edition. Wiesbaden: Aula, 1984.

[HHL] Hilgert, Joachim; Hofmann, Karl-Heinrich and Lawson, Jimmie D. *Lie Groups, Convex Cones, and Semigroups*. Oxford: Clarendon Press, 1989.

[HN] Hilgert, Joachim and Neeb, Karl-Hermann. *Lie-Gruppen und Lie-Algebren*. Braunschweig: Vieweg, 1991.

[RUD] Rudin, Walter. *Functional Analysis*. New Delhi: TATA McGraw-Hill, 1974.

[SAKU] Sakurai, Jun John. *Modern Quantum Mechanics*. N.p.: Addison-Wesley,1985.

[TVS] Schaefer, Helmut H. *Topological Vector Spaces*. New York: Springer, 1970.

[WGT] Willard, Stephen. *General Topology*. N.p.: Addison-Wesley, 1970.

[MATH] Wolfram Research, Inc. *Mathematica*. Version 2.2. Champaign, Illinois: Wolfram Research, Inc., 1994.

(N.p. means no place)

Index